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The Berwald-type linearization of generalized connections

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Abstract
We study the existence of a natural 'linearization' process for generalized connections on an affine bundle. It is shown that this leads to an affine generalized connection over a prolonged bundle, which is the analogue of what is called a connection of Berwald type in the standard theory of connections. Various new insights are being obtained in the fine structure of affine bundles over an anchored vector bundle and affineness of generalized connections on such bundles.

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1. Introduction

The notion of Berwald connection seems to have its origin in Finsler geometry. A Finsler spray generates a non-linear connection on the tangent bundle $\tau_M : TM \to M$ and the Finslerian Berwald connection represents a linearized version of this non-linear connection. It is by now well known, however, that this linearization process can be applied to any non-linear connection on the tangent bundle, resulting in a connection which is said to be 'of Berwald type'. There are different equivalent descriptions in the literature concerning this process of linearization, in particular concerning the kind of bundle on which this is taking place (see, e.g., [1, 6, 28]). We adopt the line of thinking of, for example, [6, 18], where the linear connection associated with a non-linear one on τ_M, is regarded as a connection on the pullback bundle $\tau_M^* TM \to TM$.

Much earlier, Vilms showed [29] that Berwald-type connections can be constructed starting from a non-linear connection on an arbitrary vector bundle, not necessarily a tangent bundle. In this paper, we wish to extend the notion of Berwald connection even further by investigating the following kind of generalization. In the first place, motivated by the recent interest in so-called 'generalized connections', we shall explore how the linearization idea works in that context. Generalized connections are connections on a bundle $\pi : E \to M$ over a vector bundle $\mathsf{V} \to M$ say, which is anchored in *TM* via a bundle morphism $\rho : \mathsf{V} \to TM$ (think, for example, of a Lie algebroid). Such connections show up in various fields of application: see, for example, the work of Fernandes on Poisson geometry [11], and recent applications discussed by Langerock in the fields of non-holonomic mechanics [14], sub-Riemannian geometry [15], and control theory [16]. For a general account on generalized connections, see [4] and references therein. In many applications, the bundle π will itself be a vector bundle. The second kind of generalization we wish to investigate, however, is the situation where π is an affine bundle. In fact, this paper will almost entirely deal with the case of an affine bundle, because there are lots of subtle points to be understood in such a case, while it is easy to deduce the corresponding results for vector bundles from the affine case.

An obvious motivation for paying attention to the case of affine bundles comes from the geometry of time-dependent mechanical systems. Time-dependent second-order ordinary differential equations, for example (SODEs for short), are modelled by a vector field on the first-jet bundle of a manifold which is fibred over the real numbers: this is an affine bundle over the base manifold. SODEs provide a canonically defined non-linear connection on this affine bundle and the associated linear connection has played a role in a variety of applications. In [24], we have made a quite exhaustive comparative study of different versions of this linear connection (not always called a Berwald-type connection in the literature), which were independently described by Byrnes [2], Massa and Pagani [23] and Crampin *et al* [7]. As said before, our line of approach was to view this linear connection as being defined on some pullback bundle. We observed that the differences come from a kind of 'gauge freedom' in fixing the 'time-component' of the connection. In particular, we found that there are two rather natural ways of fixing this freedom. Our present contribution is in some respect a continuation of this work. As a matter of fact, within the much more general context of generalized connections and arbitrary affine bundles, we will be led to a very clear understanding of the origin of these two competing natural constructions. The present work further links up with our recent studies of affine Lie algebroids and 'time-dependent Lagrangian systems' defined on such affine algebroids [22, 27]. Last but not least, this paper complements (and in fact was announced in) our recent analysis of affineness of generalized connections [25].

In section 2, we recall the basic features of affine bundles and generalized connections, needed for the rest of the paper. Most of section 3 is about the special case that the generalized connection on the affine bundle is itself affine, in the sense of [25]. We study the relationship between parallel transport along an admissible curve and Lie transport of vertical vectors along the horizontal lift of such a curve, and further arrive at explicit defining relations for the covariant derivative operators associated with an affine generalized connection. This paves the way to the analysis of the next section, where the idea is to conceive a notion of a Berwald-type connection associated with an arbitrary (non-linear) generalized connection. This Berwald-type connection indeed appears to be an affine generalized connection over the prolonged bundle of the original affine bundle $\pi : E \to M$. In the spirit of [6], the way the idea of Berwald-type connections is being developed comes from looking at natural ways of defining (on the pullback bundle π^*E) rules of parallel transport along horizontal and vertical curves in *E*. There appear to be two ways of defining a kind of notion of complete parallellism in the fibres of *E*; they give rise to two different Berwald-type connections which relate back to the results for time-dependent mechanics in [24]. In section 5, we further specialize to the case where the affine bundle is an affine Lie algebroid and the generalized connection is the one canonically associated with a given pseudo-SODE, as discussed already in [26]. The case of Lagrangian systems on affine Lie algebroids then is a further particular situation.

2. Basic setup

Let $\pi : E \to M$ be an affine bundle, modelled on the vector bundle $\bar{\pi} : \bar{E} \to M$. The set of all affine functions on E_m ($m \in M$), $E_m^\dagger := \mathrm{Aff}(E_m, \mathbb{R})$ is the typical fibre of a vector bundle $E^\dagger = \bigcup_{m\in M} E_m^\dagger$ over M, called the extended dual of E. In turn, the dual of $\pi^\dagger : E^\dagger \to M$, denoted by $\tilde{\pi} : \tilde{E} := (E^\dagger)^* \to M$, is a vector bundle into which both E and $\bar{E}$ can be mapped via canonical injections (see [22]). In fact, for $m \in M$, E_m is identified with the subset of elements of $\tilde{E}_m$ which map a constant function in $E_m^\dagger$ onto its value; likewise $\bar{E}_m$ can be thought of as consisting of those elements of $\tilde{E}_m$ which vanish on constant functions. We will further need the pullback bundles $\pi^*\pi : \pi^*E \to E$, $\pi^*\bar{\pi} : \pi^*\bar{E} \to E$ and $\pi^*\tilde{\pi} : \pi^*\tilde{E} \to E$. In the following, we will not make a notational distinction between a point in E and its injection in $\tilde{E}$ (and likewise for a vector in $\bar{E}$). Similarly, if for example, σ denotes a section of the affine bundle, the same symbol will be used for its injection in $\mathrm{Sec}(\tilde{\pi})$ and even for the section $\sigma \circ \pi$ of $\pi^*\tilde{\pi}$ (if one looks at a section of $\pi^*\tilde{\pi}$ as a map $\tilde{X} : E \to \tilde{E}$ such that $\tilde{\pi} \circ \tilde{X} = \pi$). We trust that the meaning will be clear from the context.

The structure of the $C^\infty(E)$-module $\mathrm{Sec}(\pi^*\tilde{\pi})$ deserves some closer inspection. The injection of E into $\tilde{E}$ provides a canonical section of $\pi^*\tilde{\pi}$, which will be denoted by $\mathcal{I}$. Furthermore, there exists a canonical map $\vartheta : \mathrm{Sec}(\pi^*\tilde{\pi}) \to \mathrm{Sec}(\pi^*\tilde{\pi})$, which can be discovered as follows (see [22]). First, within a fixed fibre $\tilde{E}_m$, every $\tilde{e}$ defines a unique number, $\lambda(\tilde{e})$ say, determined by $\lambda(\tilde{e}) = \tilde{e}(1)$, where 1 is the constant function $1 \in E_m^\dagger$ ($\lambda(\tilde{e})$ is zero when $\tilde{e}$ belongs to $\bar{E}_m$ and one when $\tilde{e} \in E_m$). If $\lambda(\tilde{e}) \neq 0$, then there exists a $e \in E_m$ such that $\tilde{e} = \lambda(\tilde{e})e$. Thus, choosing an arbitrary $a \in E_m$, we get a map $\vartheta_a : \tilde{E}_m \to \tilde{E}_m$, $\tilde{e} \mapsto \tilde{e} - \lambda(\tilde{e})a$, which actually takes values in $\bar{E}_m$, and therefore a map

$$\vartheta : \pi^*\tilde{E} \to \pi^*\bar{E} \subset \pi^*\tilde{E}, (a, \tilde{e}) \mapsto (a, \tilde{e} - \lambda(\tilde{e})a). \tag{1}$$

We will use the same notation for the extension of this map to $\mathrm{Sec}(\pi^*\tilde{\pi})$, i.e. for $\tilde{X} \in \mathrm{Sec}(\pi^*\tilde{\pi})$, $\vartheta(\tilde{X})(e) = \vartheta(\tilde{X}(e))$. It follows that every $\tilde{X} \in \mathrm{Sec}(\pi^*\tilde{\pi})$ can be written in the form

$$\tilde{X} = f_{\tilde{X}}\mathcal{I} + \vartheta(\tilde{X}) \qquad \text{with} \quad f_{\tilde{X}} \in C^\infty(E) : f_{\tilde{X}}(e) = \lambda(\tilde{X}(e)). \tag{2}$$

Clearly, if $\tilde{X} = \vartheta(\tilde{X})$ in some open neighbourhood in E, it means that $\lambda(\tilde{X}(e)) = 0$, so that $\tilde{X}(e) \in \bar{E}$ in that neighbourhood, and such a $\tilde{X}$ cannot exist at the same time in the span of $\mathcal{I}$. We conclude that locally

$$\mathrm{Sec}(\pi^*\tilde{\pi}) = \langle \mathcal{I} \rangle \oplus \mathrm{Sec}(\pi^*\bar{\pi}). \tag{3}$$

As a consequence, if $\{\bar{\sigma}_\alpha\}$ is a local basis for $\mathrm{Sec}(\bar{\pi})$, then $\{\mathcal{I}, \bar{\sigma}_\alpha\}$ is a local basis for $\mathrm{Sec}(\pi^*\tilde{\pi})$.

For any affine bundle, there exists a well-defined notion of *vertical lift* (cf [22]). The vertical lift of a vector $\bar{e} \in \bar{E}_m$ to an element of T_eE at a point $e \in E_m$ is the point $v(e, \bar{e})$ determined by the requirement that for all functions $f \in C^\infty(E)$:

$$v(e, \bar{e})f = \frac{\mathrm{d}}{\mathrm{d}t} f(e + t\bar{e})|_{t=0}. \tag{4}$$

Then, any $(e, \tilde{e}) \in \pi^*\tilde{E}$ can be vertically lifted to the point $v(e, \tilde{e})$ in the fibre T_eE over e, determined by

$$v(e, \tilde{e}) = v(e, \vartheta_e(\tilde{e})). \tag{5}$$

The final step of course is to extend this construction in the obvious way to an operation:

$$v : \mathrm{Sec}(\pi^*\tilde{\pi}) \to \mathcal{X}(E). \tag{6}$$

It follows in particular that

$$v(\mathcal{I}) = 0. \tag{7}$$

Given a vertical Q in TE, we will use the notation Q_v for the unique element in $\pi^*\bar{E}$ such that $v(Q_v) = Q$.

In what follows, there will be a role also for a second vector bundle $\tau : \mathsf{V} \to M$ which is anchored in TM by means of a linear bundle map $\varrho : \mathsf{V} \to TM$. Note that ϱ can be regarded in an obvious way also as a map (with the same symbol) from $\pi^*\mathsf{V}$ into π^*TM, by means of $\varrho(e, \mathsf{v}) = (e, \varrho(\mathsf{v}))$ for any $(e, \mathsf{v}) \in \pi^*\mathsf{V}$. The generalized connections we will be concerned with in the rest of this paper are so-called *ϱ-connections on π*. They are defined (see [4] and references therein) as follows.

Definition 1. *A ϱ-connection on π is a linear bundle map $h : \pi^*\mathsf{V} \to TE$ such that $T\pi \circ h = \varrho \circ p_{\mathsf{V}}$.*

Here, p_{V} is the projection $\pi^*\mathsf{V} \to \mathsf{V}$. In [25], we have shown that the terminology 'connection' is justified here, since a ϱ-connection can be seen, alternatively, as a splitting of some short exact sequence. For that purpose, one has to invoke the *ϱ-prolongation of π* (see e.g., [12, 17]). It is the bundle $\pi^1 : T^\varrho E \to E$ whose total space $T^\varrho E$ is the total space of the pullback bundle ϱ^*TE,

$$T^\varrho E = \{(\mathsf{v}, Q_e) \in \mathsf{V} \times TE \mid \varrho(\mathsf{v}) = T\pi(Q_e)\} \tag{8}$$

whereby the projection π^1 is the composition of the projection ϱ^1 of ϱ^*TE onto TE with the tangent bundle projection $\tau_E, \pi^1 = \tau_E \circ \varrho^1$. The vector bundle π^1 has a well-defined subbundle $\mathcal{V}^\varrho E \to E$, the vertical bundle, consisting of those elements that lie in the kernel of the projection $T^\varrho E \to \mathsf{V}$. These elements are of the form $(0, Q_e)$, where Q_e is also vertical in T_eE. We can now extend the vertical lift v to a map ${}^V : \pi^*\tilde{E} \to T^\varrho E$, by means of

$$(e, \tilde{e})^V = (0, v(e, \tilde{e})). \tag{9}$$

The point now is that a ϱ-connection on π can equivalently be seen as a splitting of the short exact sequence

$$0 \to \mathcal{V}^\varrho E \to T^\varrho E \xrightarrow{j} \pi^*\mathsf{V} \to 0 \tag{10}$$

with $j : T^\varrho E \to \pi^*\mathsf{V} : (\mathsf{v}, Q) \mapsto (\tau_E(Q), \mathsf{v})$, i.e. as a map ${}^H : \pi^*\mathsf{V} \to T^\varrho E$ such that $j \circ {}^H = \mathrm{id}_{\pi^*\mathsf{V}}$. The relation between the maps h and ^{H} is: $\varrho^1 \circ {}^H = h$. As always, we will use the same symbol for the extension of the maps h, v, ^{H} and ^{V} to sections of the corresponding bundles. As a consequence of the existence of a splitting, for any section $\mathcal{Z} \in \mathrm{Sec}(\pi^1)$, there exist uniquely determined sections $\mathsf{X} \in \mathrm{Sec}(\pi^*\tau)$ and $\bar{Y} \in \mathrm{Sec}(\pi^*\bar{\pi})$ such that

$$\mathcal{Z} = \mathsf{X}^H + \bar{Y}^V. \tag{11}$$

In fact, if $\{\mathsf{s}_a\}$ is a local basis for $\mathrm{Sec}(\tau)$ and $\{\bar{\sigma}_\alpha\}$ a basis for $\mathrm{Sec}(\bar{\pi})$, and these are interpreted as sections of $\pi^*\mathsf{V} \to E$ and $\pi^*\bar{E} \to E$, respectively, then $\left\{\mathsf{s}_a^H, \bar{\sigma}_\alpha^V\right\}$ provides a local basis for $\mathrm{Sec}(\pi^1)$.

A summary of most spaces and maps involved in the above construction is presented in the following diagram.

$$\begin{tikzcd}
 & TE \arrow[ddr, "\tau_E"] \arrow[dddd, "T\pi"] & \\
T^{\varrho}E \arrow[ur, "\varrho^1"] \arrow[drr, "\pi^1"] \arrow[dd, shift left, "j"] & & \\
 & & E \arrow[ddd, "\pi"] \\
\pi^*\mathsf{V} \arrow[uu, shift left, "H"] \arrow[uuur, "h"] \arrow[urr, "\pi^*\tau"] \arrow[dd, "p_{\mathsf{V}}"] & & \\
 & TM \arrow[dr, "\tau_M"] & \\
\mathsf{V} \arrow[ur, "\varrho"] \arrow[rr, "\tau"] & & M
\end{tikzcd}$$

For later use we list here some coordinate expressions. Let x^i denote coordinates on M and y^α fibre coordinates on E with respect to some local frame $(e_0; \{\bar{e}_\alpha\})$ for $\mathrm{Sec}(\pi)$. For each $\sigma \in \mathrm{Sec}(\pi)$ with local representation $\sigma(x) = e_0(x) + \sigma^\alpha(x)\bar{e}_\alpha(x)$, the maps

$$e^0(\sigma)(x) = 1 \quad \forall x \qquad e^\alpha(\sigma)(x) = \sigma^\alpha(x) \tag{12}$$

define an induced basis for $\mathrm{Sec}(\pi^\dagger)$. Remark that e^0 coincides, in each fibre, with the constant function 1 and thus has a global character. We will denote by (e_0, e_α) the basis of $\mathrm{Sec}(\tilde{\pi})$ dual to the basis (12). Since sections of $\tilde{\pi}$ can be regarded also as (basic) sections of $\pi^*\tilde{\pi}$, (e_0, e_α) can serve at the same time as a local basis for $\mathrm{Sec}(\pi^*\tilde{\pi})$. Hence, every $\tilde{X} \in \mathrm{Sec}(\pi^*\tilde{\pi})$ can be represented in the form $\tilde{X} = \tilde{X}^0(x, y)e_0 + \tilde{X}^\alpha(x, y)e_\alpha$. But more interestingly, with the use of the canonical section $\mathcal{I}$, we have

$$\mathcal{I} = e_0 + y^\alpha e_\alpha \qquad \tilde{X} = \tilde{X}^0\mathcal{I} + (\tilde{X}^\alpha - \tilde{X}^0 y^\alpha)e_\alpha. \tag{13}$$

If we use v^a for the fibre coordinates of τ and $\{\mathsf{e}_a\}$ for the corresponding local basis, then a given anchor map ϱ has coordinate representation $\varrho : (x^i, \mathsf{v}^a) \mapsto \varrho^i_a(x)\mathsf{v}^a \frac{\partial}{\partial x^i}$. In [25], we have shown that a natural choice for a local basis of sections of the prolonged bundle $\pi^1 : T^\varrho E \to E$ is given by the following: for each $e \in E$, if x are the coordinates of $\pi(e) \in M$,

$$\mathcal{X}_a(e) = \left(\mathsf{e}_a(x),\ \varrho^i_a(x)\frac{\partial}{\partial x^i}\bigg|_e\right) \qquad \mathcal{V}_\alpha(e) = \left(0,\ \frac{\partial}{\partial y^\alpha}\bigg|_e\right). \tag{14}$$

The dual basis for $\mathrm{Sec}(\pi^{1*})$ is denoted by $\{\mathcal{X}^a, \mathcal{V}^\alpha\}$. A general section of the prolonged bundle can be represented locally in the form

$$\mathcal{Z} = \mathsf{z}^a(x, y)\mathcal{X}_a + Z^\alpha(x, y)\mathcal{V}_\alpha. \tag{15}$$

Suppose that, in addition, we have a ϱ-connection on π at our disposal. As in [25], the local expressions of h and ^{H} then are

$$h(x^i, y^\alpha, \mathsf{v}^a) = \left(x^i, y^\alpha, \rho^i_a(x)\mathsf{v}^a, -\Gamma^\alpha_a(x, y)\mathsf{v}^a\right) \tag{16}$$

and

$$(x^i, y^\alpha, \mathsf{v}^a)^H = \left((x^i, \mathsf{v}^a), \mathsf{v}^a\left(\rho^i_a\frac{\partial}{\partial x^i} - \Gamma^\alpha_a\frac{\partial}{\partial y^\alpha}\right)\right). \tag{17}$$

We can now easily give a local basis for the horizontal sections of π^1, which is given by

$$\mathcal{H}_a = \mathsf{e}_a{}^H = \mathcal{X}_a - \Gamma^\alpha_a(x, y)\mathcal{V}_\alpha. \tag{18}$$

An adapted representation of the section (15) then becomes

$$\mathcal{Z} = \mathsf{z}^a\mathcal{H}_a + \big(Z^\alpha + \mathsf{z}^a\Gamma^\alpha_a\big)\mathcal{V}_\alpha. \tag{19}$$

We end this section with the following remark about a generalized notion of tangent map between ϱ-prolongations. Suppose that two vector bundles $\mathsf{V}_1 \to M_1$ and $\mathsf{V}_2 \to M_2$ with anchors ϱ_1 and ϱ_2 (respectively) are given, together with two arbitrary fibre bundles $P_1 \to M_1$ and $P_2 \to M_2$. Suppose further that $F : P_1 \to P_2$ is a bundle map over some $f : M_1 \to M_2$ and that $\mathsf{f} : \mathsf{V}_1 \to \mathsf{V}_2$ is a vector bundle morphism over the same f, satisfying $Tf \circ \varrho_1 = \varrho_2 \circ \mathsf{f}$. Then, we can define a map

$$T^{\varrho_1,\varrho_2}F : T^{\varrho_1}P_1 \to T^{\varrho_2}P_2, (\mathsf{v}_1, X_1) \mapsto (\mathsf{f}(\mathsf{v}_1), TF(X_1)). \tag{20}$$

3. Parallel transport and Lie transport for affine generalized connections

Let $h : \pi^*\mathsf{V} \to TE$ be a ϱ-connection on an affine bundle $\pi : E \to M$. The affine structure of E can be represented by the map $\Sigma : E \times_M \bar{E} \to E$, $\Sigma(e, \bar{e}) = e + \bar{e}$ ($e \in E_m, \bar{e} \in \bar{E}_m$). The ϱ-connection h on π is said to be *affine*, if there exists a linear ϱ-connection $\bar{h}$ on $\bar{\pi}$ such that $\forall e \in E_m, \bar{e} \in \bar{E}_m$ and $\mathsf{v} \in \mathsf{V}_m$

$$h(e + \bar{e}, \mathsf{v}) = T\Sigma_{(e,\bar{e})}(h(e, \mathsf{v}), \bar{h}(\bar{e}, \mathsf{v})). \tag{21}$$

The connection coefficients of an affine connection are of the form $\Gamma^\alpha_a(x, y) = \Gamma^\alpha_{a0}(x) + \Gamma^\alpha_{a\beta}(x)y^\beta$. We have shown in [25] that an affine ϱ-connection on π can equivalently be seen as a pair $(\nabla, \bar{\nabla})$ where $\bar{\nabla}$ is the covariant derivative operator corresponding to the linear $\bar{h}$ and $\nabla : \mathrm{Sec}(\tau) \times \mathrm{Sec}(\pi) \to \mathrm{Sec}(\bar{\pi})$ is an operator which is $\mathbb{R}$-linear in its first argument and has the properties

$$\nabla_{f\mathsf{s}}\sigma = f\nabla_{\mathsf{s}}\sigma \qquad \nabla_{\mathsf{s}}(\sigma + f\bar{\sigma}) = \nabla_{\mathsf{s}}\sigma + f\bar{\nabla}_{\mathsf{s}}\bar{\sigma} + \varrho(\mathsf{s})(f)\bar{\sigma} \tag{22}$$

for all $\mathsf{s} \in \mathrm{Sec}(\tau), \sigma \in \mathrm{Sec}(\pi), \bar{\sigma} \in \mathrm{Sec}(\bar{\pi})$ and $f \in C^\infty(M)$. ∇ is defined via the so-called connection map $K : T^\varrho E \to \bar{E}$, which maps (v, Q_e) to the vertical tangent vector $(\varrho^1 - h \circ j)(\mathsf{v}, \mathsf{Q_e})$ thought of as an element of $\bar{E}$,

$$\nabla_{\mathsf{s}}\sigma(m) = K(\mathsf{s}(m), T\sigma(\varrho(\mathsf{s}(m)))). \tag{23}$$

Technically, $K = p_{\bar{E}} \circ {}_v \circ (\varrho^1 - h \circ j)$.

Let us briefly summarize the discussion of parallel transport for affine ϱ-connections, as developed in [25]. A curve in $\pi^*\mathsf{V}$ is a couple (ψ, c), where ψ is a curve in E and c a curve in V with the properties that the projected curves on M coincide: $\psi_M = c_M$ (in taking a curve in $\pi^*\mathsf{V}$ we will suppose that $I = [a, b] \subset \mathrm{Dom}(\psi, \mathsf{c})$ is an interval in $\mathrm{Dom}(\psi) \cap \mathrm{Dom}(\mathsf{c})$). The *horizontal lift of the curve* (ψ, c) is a curve $(\psi, \mathsf{c})^H$ in $T^\varrho E$, determined by

$$(\psi, \mathsf{c})^H : u \mapsto (\mathsf{c}(u), h(\psi(u), \mathsf{c}(u))) \qquad \text{for all } u \in I. \tag{24}$$

We also say that ψ in E is *a horizontal lift of* c, and write $\psi = c^h$, if $(\psi, \mathsf{c})^H$ is a ϱ^1-admissible curve. Since by construction $\pi^1 \circ (\psi, \mathsf{c})^H = \psi$, this means that

$$\dot{\psi}(u) = \varrho^1 \circ (\psi, \mathsf{c})^H(u) = h(\psi(u), \mathsf{c}(u)) \tag{25}$$

and automatically implies that c must be ϱ-admissible, since $\dot{c}_M = \dot{\psi}_M = T\pi \circ \dot{\psi} = T\pi \circ h(\psi, \mathsf{c}) = \varrho \circ \mathsf{c}$. Given c, with $c_M(a) = m$ say, we will write c^h_e for the unique solution of (25) passing through the point $e \in E_m$ at $u = a$ (i.e., $c^h_e(a) = e$) and denote the lift $\big(c^h_e, \mathsf{c}\big)^H$ by $\dot{c}^H_e$, where, of course, the 'dot' merely refers to the fact that $\varrho^1\big(\dot{c}^H_e\big) = \dot{c}^h_e$. Curves of the form $\dot{c}^H_e$ are ϱ^1-admissible by construction.

Let s be a section of $\mathsf{V} \to M$, which we regard as a section of $\pi^*\mathsf{V} \to E$ via the composition with π. In that sense, we can talk about the vector field $h(\mathsf{s}) \in \mathcal{X}(E)$, which has the following interesting characteristics.

Lemma 1. *For any $\mathsf{s} \in \mathrm{Sec}(\tau)$, the vector field $h(\mathsf{s})$ on E has the property that all its integral curves are horizontal lifts of ϱ-admissible curves in V.*

Proof. Let γ_e denote an integral curve of $h(\mathsf{s})$ through the point e. Since $h(\mathsf{s})$ is π-related to $\varrho(\mathsf{s}) \in \mathcal{X}(M)$, $\pi \circ \gamma_e$ then is an integral curve of $\varrho(\mathsf{s})$ through $m = \pi(e)$, which we shall call c_m. Obviously, $\mathsf{c} = \mathsf{s}(c_m)$ now is a ϱ-admissible curve in V and we have for all u in its domain,

$$h(\mathsf{s})\big(c_e^h(u)\big) = h\big(c_e^h(u), \mathsf{s}\big(\pi\left(c_e^h(u)\right)\big)\big) = h\big(c_e^h(u), \mathsf{c}(u)\big) = \dot{c}_e^h(u) \tag{26}$$

which shows that $\gamma_e = c_e^h$. □

So far, the above characterization of a horizontal lift applies to any ϱ-connection on π. If, in particular, the connection is *affine*, then we can express the definition of a horizontal lift also in terms of the operator ∇. Indeed, for any ϱ-admissible curve c in V and any curve ψ with $c_M = \psi_M$, we can define a new curve $\nabla_{\mathsf{c}}\psi$ by means of

$$\nabla_{\mathsf{c}}\psi(u) := K(\mathsf{c}(u), \dot{\psi}(u)) = (\dot{\psi}(u) - h(\psi(u), \mathsf{c}(u)))_v. \tag{27}$$

Obviously, $\psi = c_e^h$ iff $\psi(a) = e$ and $\nabla_{\mathsf{c}}\psi = 0$ for all $u \in I$.

Putting $c_M(b) = m_b$, the point $c_e^h(b) \in E_{m_b}$ is called the *parallel translate of e along* c. It is instructive to see in detail how one can get an affine action between the affine fibres of E. Take $e_1, e_2 \in E_m$ and consider the horizontal lifts $c_{e_1}^h$ and $c_{e_2}^h$. Denote the difference $e_1 - e_2$ by $\bar{e} \in \bar{E}_m$ and put $\bar{\eta}_{\bar{e}} := c_{e_1}^h - c_{e_2}^h$. As the subscript indicates, $\bar{\eta}_{\bar{e}}$ is a curve in $\bar{E}$ starting at $\bar{e}$. From the action of ∇ on curves (see, e.g., [4, 25]), it easily follows that

$$\nabla_{\mathsf{c}} c_{e_1}^h(u) - \nabla_{\mathsf{c}} c_{e_2}^h(u) = \bar{\nabla}_{\mathsf{c}}\bar{\eta}_{\bar{e}}(u) \qquad \text{for all } u \in I. \tag{28}$$

Since both $c_{e_1}^h$ and $c_{e_2}^h$ are solutions of the equation $\nabla_{\mathsf{c}}\psi = 0$, $\bar{\eta}_{\bar{e}}$ must be the unique solution of the initial value problem $\bar{\nabla}_{\mathsf{c}}\bar{\eta} = 0$, $\bar{\eta}(a) = \bar{e}$, i.e. the unique $\bar{h}$-horizontal lift through $\bar{e}$. Therefore, *the difference between the ∇-parallel translates of e_1 and e_2 along c is the $\bar{\nabla}$-parallel translate of $e_1 - e_2$ along* c. In fact, this property is necessary and sufficient for the connection to be *affine*. From now on we will use the notation $c_{\bar{e}}^{\bar{h}}$ for $c_{e_1}^h - c_{e_2}^h$.

Proposition 1. *A ϱ-connection h on π is affine if and only if there exists a linear ϱ-connection $\bar{h}$ on $\bar{\pi}$, such that for all admissible curves c and for any two points $e_1, e_2 \in E_m (m = c_M(a))$ the difference $c_{e_1}^h(u) - c_{e_2}^h(u)$ is the $\bar{\nabla}$-parallel translate of $e_1 - e_2$ along* c.

Proof. The proof in one direction has already been given. For the converse, suppose that a *linear* connection $\bar{h}$ exists, having the above properties. It suffices to show that $\bar{h}$ is related to h by means of (21). Choosing a $\mathsf{v} \in \mathsf{V}_m$ arbitrarily, we take a ϱ-admissible curve c, that passes through it (for $u = a$) and consider its h-horizontal lift c_e^h through e and its $\bar{h}$-horizontal lift through $\bar{e}$. By assumption we know that $c_{e+\bar{e}}^h - c_e^h$ is the $\bar{h}$-horizontal lift of c through $\bar{e}$, i.e.

$$\Sigma\big(c_e^h(u), c_{\bar{e}}^{\bar{h}}(u)\big) = c_{e+\bar{e}}^h(u) \qquad \text{for all } u \in I. \tag{29}$$

Taking the derivative of this expression at $u = a$, we get

$$T\Sigma_{(e,\bar{e})}\big(\dot{c}_e^h(a), \dot{c}_{\bar{e}}^{\bar{h}}(a)\big) = \dot{c}_{e+\bar{e}}^h(a). \tag{30}$$

In view of (25) and its analogue for $\bar{h}$, this is indeed what we wanted to show. □

Let us introduce corresponding 'flow-type' maps. For that purpose, it is convenient to use (temporarily) the more accurate notation $c^h_{a,e}$ for the horizontal lift which passes through e at $u = a$. Putting

$$\phi^h_{u,a}(e) = c^h_{a,e}(u) \qquad u \in [a, b] \tag{31}$$

the result of the preceding proposition can equivalently be expressed as

$$\phi^h_{u_2,u_1}(e + \bar{e}) = \phi^h_{u_2,u_1}(e) + \phi^{\bar{h}}_{u_2,u_1}(\bar{e}) \tag{32}$$

i.e. $\phi^h_{u_2,u_1} : E_{c_M(u_1)} \to E_{c_M(u_2)}$ is an affine map with linear part $\phi^{\bar{h}}_{u_2,u_1}$. Its tangent map, therefore, can be identified with its linear part. As a result, when we consider Lie transport of vertical vectors in TE along the curve $c^h_{a,e}$, in the case of an affine connection, the image vectors will come from the parallel translate associated with the linear connection $\bar{h}$. Indeed, putting $Y_a = v(e, \bar{e})$ and defining its Lie translate from a to b as $Y_b = T\phi^h_{b,a}(Y_a)$, we have for each $g \in C^\infty(E)$,

$$\begin{aligned} Y_b(g) = Y_a\big(g \circ \phi^h_{b,a}\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(g \circ \phi^h_{b,a}(e + t\bar{e})\big)_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(g\big(\phi^h_{b,a}(e) + t\phi^{\bar{h}}_{b,a}(\bar{e})\big)\big)_{t=0} = v\big(c^h_{a,e}(b), c^{\bar{h}}_{a,\bar{e}}(b)\big)g \end{aligned}$$

where the transition to the last line requires affineness of the connection. It follows that in the affine case, the Lie translate of $Y_a = v(e, \bar{e})$ to b is given by

$$Y_b = v\big(c^h_{a,e}(b), c^{\bar{h}}_{a,\bar{e}}(b)\big). \tag{33}$$

At this point, it is appropriate to make a few more comments about the general idea of Lie transport. If (on an arbitrary manifold) Y is a vector field along an integral curve of some other vector field X, and we therefore have a genuine (local) flow ϕ_s at our disposal, then the Lie derivative of Y with respect to X is defined to be

$$\mathcal{L}_X Y(u) = \frac{\mathrm{d}}{\mathrm{d}s}(T\phi_{-s}(Y(s + u)))_{s=0}. \tag{34}$$

As shown for example in [8] (p 68), the requirement $\mathcal{L}_X Y = 0$, subject to some initial condition, $Y(0) = Y_0$ say, then uniquely determines a vector field Y along an integral curve of X in such a way that $Y(u)$ is obtained by Lie transport of Y_0. The description of Lie transport, therefore, is more direct when we are in the situation of an integral curve of a vector field.

Lemma 1, unfortunately, does not create such a situation for us because, when an admissible curve c is given, together with one of its horizontal lifts c^h_e, it does not provide us with a way of constructing a vector field which has c^h_e as one of its integral curves. The complication for constructing such a vector field primarily comes from the fact that the differential equations (25) which define c^h_e are non-autonomous. The usual way to get around this problem is to make the system autonomous by adding on extra dimension. A geometrical way of achieving this here, which takes into account that c^h_e in the first place has to be a curve projecting onto c_M, is obtained by passing to the pullback bundle $c^*_M E \to I \subset \mathbb{R}$. We introduce the notation $c^1_M : c^*_M E \to E, (u, e) \mapsto (c_M(u), e)$, and likewise $\bar{c}^1_M : c^*_M \bar{E} \to \bar{E}, (u, \bar{e}) \mapsto (c_M(u), \bar{e})$. With the help of these maps, we can single out vector fields $\Lambda_c \in \mathcal{X}(c^*_M E)$ and $\bar{\Lambda}_c \in \mathcal{X}(c^*_M \bar{E})$ as follows.

Proposition 2. *For any ϱ-connection h on π and given ϱ-admissible curve c in V, there exists a unique vector field Λ_c on $c^*_M E$ that projects on the coordinate vector field on $\mathbb{R}$ and is such that*

$$Tc^1_M(\Lambda_c(u, e)) = h\big(c^1_M(u, e), \mathsf{c}(u)\big) \tag{35}$$

for all $(u, e) \in c_M^* E$. Likewise, if the connection is affine with linear part $\bar h$, there exists a unique vector field $\bar\Lambda_c$ on $c_M^* \bar E$ that projects on the coordinate vector field on $\mathbb{R}$ and is such that

$$T\bar c_M^1(\bar\Lambda_c(u, \bar e)) = \bar h\big(\bar c_M^1(u, \bar e), \mathsf{c}(u)\big) \tag{36}$$

for all $(u, \bar e) \in c_M^* \bar E$.

Proof. The proof is analogous for both cases; we prove only the first. Let u denote the coordinate on $\mathbb{R}$ and y^α the fibre coordinates of some $e \in (c_M^* E)_u$. Representing the given curve as $\mathsf{c} : u \mapsto (x^i(u), \mathsf{c}^a(u))$ and putting $\Lambda_c(u, e) = \frac{\partial}{\partial u}\big|_{(u,e)} + Y^\alpha(u, e)\frac{\partial}{\partial y^\alpha}\big|_{(u,e)}$, one finds that

$$T c_M^1(\Lambda_c(u, e)) = (x^i(u), y^\alpha; \dot x^i(u), Y^\alpha(u, e)). \tag{37}$$

On the other hand, $h\big(c_M^1(u, e), \mathsf{c}(u)\big) = \big(x^i(u), y^\alpha; \mathsf{c}^a(u)\varrho_a^i(c_M(u)), -\mathsf{c}^a(u)\Gamma_a^\alpha(c_M(u), e)\big)$. Identification of the two expressions gives that $Y^\alpha(u, e) = -\mathsf{c}^a(u)\Gamma_a^\alpha(c_M(u), e)$. □

Note that, at points where c does not lie in the kernel of ϱ, the requirement (35) by itself would imply that Λ_c projects on $\mathrm{d}/\mathrm{d}u$. An interesting point is that this will generically also be satisfied automatically if we look at horizontality on $T^\varrho E$ rather than on *TE*, i.e. horizontality in the sense of (24). To see this, observe first that we can use the bundle map $\mathsf{f} : T\mathbb{R} \to \mathsf{V}, U\frac{\mathrm{d}}{\mathrm{d}u}\big|_u \mapsto U\mathsf{c}(u)$ over $c_M : I \to M$, to obtain, in accordance with (20), the following extended notion of a tangent map:

$$T^\varrho c_M^1 : T(c_M^* E) \to T^\varrho E \qquad \lambda \mapsto \big(\mathsf{f}(T\tau_{\mathbb{R}}\lambda), T c_M^1(\lambda)\big). \tag{38}$$

Here $\tau_{\mathbb{R}}$ is the bundle projection of $c_M^* E \to \mathbb{R}$. $T^\varrho c_M^1$ is well defined, since $T\tau_{\mathbb{R}}\lambda$ is of the form $U\frac{\mathrm{d}}{\mathrm{d}u}\big|_u$ and $T\pi \circ T c_M^1(\lambda) = T\big(\pi \circ c_M^1\big)(\lambda) = T(c_M \circ \tau_{\mathbb{R}})(\lambda) = T c_M \circ T\tau_{\mathbb{R}}(\lambda) = T c_M\big(U\frac{\mathrm{d}}{\mathrm{d}u}\big|_u\big) = U\dot c_M(u) = U\varrho(\mathsf{c}(u)) = \varrho\big(\mathsf{f}\big(U\frac{\mathrm{d}}{\mathrm{d}u}\big|_u\big)\big) = \varrho(\mathsf{f}(T\tau_{\mathbb{R}}\lambda))$. Note that this remains true also at points where c lies in the kernel of ϱ. Now, Λ_c can be defined as the unique vector field on $c_M^* E$ for which

$$T^\varrho c_M^1(\Lambda_c(u, e)) = \big(c_M^1(u, e), \mathsf{c}(u)\big)^H \qquad \text{for all } (u, e) \in c_M^* E. \tag{39}$$

Indeed, the second component of this equality is just the condition (35) again, whereas the first component says that $U\mathsf{c}(u) = \mathsf{c}(u)$ and thus implies $U = 1$ (except, of course, for the trivial case where c is of the form $\mathsf{c}(u) = (m_0, 0)$).

We will look now at the integral curves of Λ_c and $\bar\Lambda_c$. Since Λ_c projects on the coordinate field on $\mathbb{R}$, the integral curves are essentially sections of $c_M^* E \to \mathbb{R}$. Let γ_e denote the integral curve going through (a, e) at time $u = a$, so that

$$\dot\gamma_e(u) = \Lambda_c(\gamma_e(u)) \qquad \forall u \in I' \tag{40}$$

where I' is some interval, possibly smaller than the domain I of c. In a similar way, we will write $\bar\gamma_{\bar e}$ for the integral curve of $\bar\Lambda_c$ through $(a, \bar e)$.

Proposition 3. *For any ϱ-connection on π, the curve $c_M^1 \circ \gamma_e$ is the h-horizontal lift of c through e. Likewise, if the connection is affine with linear part $\bar h$, the curve $\bar c_M^1 \circ \bar\gamma_{\bar e}$ is the $\bar h$-horizontal lift of c through $\bar e$.*

Proof. Again, we will prove only the first statement. $c_M^1 \circ \gamma_e$ is a curve in E projecting on the curve c_M in M. Using (40) and (35) we find

$$\frac{\mathrm{d}}{\mathrm{d}u}\big(c_M^1 \circ \gamma_e\big)(u) = T c_M^1(\dot\gamma_e(u)) = T c_M^1(\Lambda_c(\gamma_e(u))) = h\big(c_M^1 \circ \gamma_e(u), c(u)\big) \tag{41}$$

which shows that $c_M^1 \circ \gamma_e = c_e^h$. □

We now proceed to look at Lie transport along integral curves of the vector field Λ_c on c_M^*E. It is clear that $c_M^*E \to \mathbb{R}$ is an affine bundle modelled on the vector bundle $c_M^*\bar{E} \to \mathbb{R}$. As we know from section 2 then, there is a vertical lift map, which we will denote by $v_{c_M^*E}$ which maps elements of $(c_M^*E \times_{\mathbb{R}} c_M^*\bar{E})$ to vertical vectors of $T(c_M^*E)$. We will consider Lie transport of such vertical vectors along integral curves of Λ_c. Starting from $e \in E_{c_M(a)}$, $\bar{e} \in \bar{E}_{c_M(a)}$ and putting $\Upsilon_{e,\bar{e}}(a) = v_{c_M^*E}((a,e),(a,\bar{e}))$, we know that the condition $\mathcal{L}_{\Lambda_c}\Upsilon_{e,\bar{e}} = 0$ uniquely defines a vector field along the integral curve $\gamma_{(a,e)}$ of Λ_c ($\gamma_{(a,e)}(a) = (a,e)$) which takes the (vertical) value $\Upsilon_{e,\bar{e}}(a)$ at the point (a,e). As said before, the value of $\Upsilon_{e,\bar{e}}$ at any later u is the Lie translate of $\Upsilon_{e,\bar{e}}(a)$, that is to say, we have $\Upsilon_{e,\bar{e}}(u) = T\phi_{u,a}(\Upsilon_{e,\bar{e}}(a))$, where $\phi_{u,a}$ refers to the flow of Λ_c ($\phi_{a,a}$ is the identity). It is interesting to observe here that $\Upsilon_{e,\bar{e}}$ is directly related to the Lie translate we discussed before, of vertical vectors on E along the horizontal lift $c_{a,e}^h$. To be precise, with $Y_{e,\bar{e}}(a) = Tc_M^1(\Upsilon_{e,\bar{e}}(a)) = v(e,\bar{e})$ and defining $Y_{e,\bar{e}}(u)$ to be $T\phi_{u,a}^h(Y_{e,\bar{e}}(a))$ as before, we have (at any later time u in the domain of $\gamma_{(a,e)}$)

$$Tc_M^1(\Upsilon_{e,\bar{e}}(u)) = Y_{e,\bar{e}}(u). \tag{42}$$

Indeed, it follows from proposition 3 (using here again the somewhat more accurate notation which takes the 'initial time' a into account), that $c_M^1 \circ \phi_{u,a} = \phi_{u,a}^h \circ c_M^1$. Therefore, we have

$$\begin{aligned} Tc_M^1(\Upsilon_{e,\bar{e}}(u)) &= Tc_M^1(T\phi_{u,a}(\Upsilon_{e,\bar{e}}(a))) = T\phi_{u,a}^h\big(Tc_M^1(\Upsilon_{e,\bar{e}}(a))\big) \\ &= T\phi_{u,a}^h(Y_{e,\bar{e}}(a)) = Y_{e,\bar{e}}(u). \end{aligned}$$

The case of an *affine* ϱ-connection is of special interest. The affine nature of the maps $c_e^h(u)$, for fixed u, implies via proposition 3 that the flow maps of Λ_c are also affine, or expressed differently that

$$\gamma_{e+\bar{e}}(u) = \gamma_e(u) + \bar{\gamma}_{\bar{e}}(u). \tag{43}$$

We have shown already that in the affine case: $Y_{e,\bar{e}}(u) = v\big(c_e^h(u), c_{\bar{e}}^{\bar{h}}(u)\big)$. The translation of this result via the relation (42) means that we have

$$\Upsilon_{e,\bar{e}}(u) = v_{c_M^*E}(\gamma_e(u), \bar{\gamma}_{\bar{e}}(u)). \tag{44}$$

Summarizing the more interesting aspects of what we have observed above, we can make the following statement.

Proposition 4. *For an arbitrary ϱ-connection on π, Lie transport of vertical vectors on E along the horizontal lift c_e^h is equivalent to Lie transport of vertical vectors on c_M^*E along integral curves of the vector field Λ_c. In the particular case that the connection is affine, the translates in both cases are compatible with the affine nature of the flow maps between fixed fibres.*

The next result concerns an important property of the covariant derivative operators ∇ and $\bar{\nabla}$ which become available when the ϱ-connection is affine. The preceding considerations about vector fields Λ_c on c_M^*E will help to prove it in a purely geometrical way. The reader may wish to skip this rather technical proof and pass to the remark immediately following it.

Proposition 5. *Let h be an affine ϱ-connection. For all $\mathsf{s} \in Sec(\tau)$, $\bar{\sigma} \in Sec(\bar{\pi})$ and $\sigma \in Sec(\pi)$, the brackets $[h\mathsf{s}, v\bar{\sigma}]$ and $[h\mathsf{s}, v\sigma]$ of vector fields on E are vertical and we have*

$$\bar{\nabla}_{\mathsf{s}}\bar{\sigma} = [h\mathsf{s}, v\bar{\sigma}]_v \qquad \nabla_{\mathsf{s}}\sigma = [h\mathsf{s}, v\sigma]_v. \tag{45}$$

Proof. We start with the bracket $[h\mathsf{s}, v\bar{\sigma}]_v$. Since the vector fields under consideration are π-related to $\varrho(\mathsf{s})$ and the zero vector field on M, respectively, their Lie bracket is π-related to $[\varrho(\mathsf{s}), 0] = 0$ and is therefore vertical. If we project it down to $\bar{E}$, strictly speaking by taking $([h\mathsf{s}, v\bar{\sigma}](e))_v \in E \times_M \bar{E}$ with $e \in E_m$ and looking at the second component, we obtain an element of $\bar{E}_m$ which does not depend on the fibre coordinates of e (as we can see instantaneously by thinking of coordinate expressions). In other words, $[h\mathsf{s}, v\bar{\sigma}]_v$ gives rise to a section of $\bar{\pi}$ which has the following properties: for all $f \in C^\infty(M)$

$$[h(f\mathsf{s}), v\bar{\sigma}]_v = f[h\mathsf{s}, v\bar{\sigma}]_v \qquad [h\mathsf{s}, v(f\bar{\sigma})]_v = f[h\mathsf{s}, v\bar{\sigma}]_v + \varrho(\mathsf{s})(f)\bar{\sigma}. \tag{46}$$

These are precisely the properties of the covariant derivative operator $\bar{\nabla}_\mathsf{s}\bar{\sigma}$, from which it follows that $L(\mathsf{s}, \bar{\sigma}) = \bar{\nabla}_\mathsf{s}\bar{\sigma} - [h\mathsf{s}, v\bar{\sigma}]_v$ is tensorial in s and $\bar{\sigma}$.

To prove that L is actually zero, we will use a rather subtle argument, which is based on the following considerations of a quite general nature. If $L(\mathsf{s}, \bar{\sigma})$ is tensorial, so that for all m, $L(\mathsf{s}, \bar{\sigma})(m)$ depends on $\mathsf{s}(m)$ and $\bar{\sigma}(m)$ only, then for each curve c_M in M, there exists a corresponding operator $l(\mathsf{r}, \bar{\eta})$, acting on arbitrary sections along the curve c_M, which is completely determined by the property: if $\mathsf{s} \in \mathrm{Sec}(\tau)$, $\bar{\sigma} \in \mathrm{Sec}(\bar{\pi})$ and we put $\mathsf{r} = \mathsf{s}|_{c_M}$, $\bar{\eta} = \bar{\sigma}|_{c_M}$, then $l(\mathsf{r}, \bar{\eta})(u) = L(\mathsf{s}, \bar{\sigma})(c_M(u))$. In turn, the value of $L(\mathsf{s}, \bar{\sigma})(m)$ at an arbitrary point m can be computed by choosing an arbitrary curve c_M through m ($c_M(0) = m$ say), selecting sections r and $\bar{\eta}$ along c_M for which $\mathsf{r}(0) = \mathsf{s}(m)$ and $\bar{\eta}(0) = \bar{e} = \bar{\sigma}(m)$, and then computing $l(\mathsf{r}, \bar{\eta})(0)$.

We apply this general idea in the following way. Starting from an arbitrary $e \in E_m$, we know from lemma 1 that the integral curve of $h(\mathsf{s})$ through e is the horizontal lift c_e^h of some admissible curve c through $\mathsf{s}(m)$ (here the 'initial time' a is taken to be zero). Take r to be this curve c (with projection c_M) and choose $\bar{\eta}$ to be the curve $c_{\bar{e}}^{\bar{h}}$. Then,

$$l(\mathsf{r}, \bar{\eta})(0) = \bar{\nabla}_\mathsf{c} c_{\bar{e}}^{\bar{h}}(0) - \left(\mathcal{L}_{h\mathsf{s}} v\left(c_e^h, c_{\bar{e}}^{\bar{h}}\right)\right)_v(0). \tag{47}$$

But $\bar{\nabla}_\mathsf{c} c_{\bar{e}}^{\bar{h}}$ is zero by construction. Concerning the second term, we observe that $h\mathsf{s}$ is c_M^1-related to Λ_c, by definition of Λ_c. Also $v\left(c_e^h, c_{\bar{e}}^{\bar{h}}\right)$ is c_M^1-related to the vector field $\Upsilon_{e,\bar{e}}(u)$ along the integral curve γ_e of Λ_c through the point $(0, e) \in c_M^* E$ (see (42)). But we know that $\mathcal{L}_{\Lambda_c}\Upsilon_{e,\bar{e}} = 0$, so that in particular $\left(\mathcal{L}_{h\mathsf{s}} v\left(c_e^h, c_{\bar{e}}^{\bar{h}}\right)\right)_v(0) = 0$. It follows that $\bar{\nabla}_\mathsf{s}\bar{\sigma} = [h\mathsf{s}, v\bar{\sigma}]_v$.

For the second part, we should specify in the first place what is meant by $v(\sigma)$: any $\sigma \in \mathrm{Sec}(\pi)$ can be thought of as a section of $\bar{\pi}$ and then $v(\sigma)(e) = v(e, \vartheta_e(\sigma(\pi(e))))$. Making use of the canonical section $\mathcal{I}$ of $\pi^*\bar{\pi}$, we can write in fact that $v(\sigma) = v(\sigma - \mathcal{I})$, where $\sigma - \mathcal{I} \in \mathrm{Sec}(\pi^*\bar{\pi})$. It is clear that $[h\mathsf{s}, v\sigma]$ is vertical again, and we find the properties: $\forall f \in C^\infty(M)$,

$$\begin{aligned}
[h(f\mathsf{s}), v\sigma]_v &= f[h\mathsf{s}, v\sigma]_v \\
[h\mathsf{s}, v(\sigma + f\bar{\sigma})]_v &= [h\mathsf{s}, v(\sigma - \mathcal{I}) + f v(\bar{\sigma})]_v \\
&= [h\mathsf{s}, v\sigma]_v + f[h\mathsf{s}, v\bar{\sigma}]_v + \varrho(\mathsf{s})(f)\bar{\sigma} \\
&= [h\mathsf{s}, v\sigma]_v + f\bar{\nabla}_\mathsf{s}\bar{\sigma} + \varrho(\mathsf{s})(f)\bar{\sigma}.
\end{aligned} \tag{48}$$

Again, these are the characterizing properties of the covariant derivative $\nabla_\mathsf{s}\sigma$. It follows that the operator $L(\mathsf{s}, \sigma) = \bar{\nabla}_\mathsf{s}\sigma - [h\mathsf{s}, v\sigma]_v$ is linear in s and affine in σ. This is the analogue, when there are affine components involved, of an operator L being tensorial. The rest of the reasoning follows the same pattern as before. This time, starting from an arbitrary $e \in E_m$ and an integral curve c_e^h of $h(\mathsf{s})$ through e, we put $\bar{e} = \sigma(\pi(e)) - e$ and choose the curves $\bar{\eta} = c_{\bar{e}}^{\bar{h}}$ in $\bar{E}$ and $\eta = c_e^h + \bar{\eta}$ in E to obtain a section of π along c_e^h which has $\sigma(\pi(e))$ as initial value. □

Remark 1. A more direct, but perhaps geometrically less appealing proof, consists in verifying the statements of proposition 5 by a coordinate calculation. We have, for $\mathsf{s} = \mathsf{s}^a(x)\mathsf{e}_a$ and $\sigma = e_0 + \sigma^\alpha(x)\bar{e}_\alpha$,

$$\nabla_\mathsf{s}\sigma = [h\mathsf{s}, v\sigma]_v = \left(\rho_a^i \frac{\partial \sigma^\alpha}{\partial x^i} + \Gamma_{a0}^\alpha(x) + \Gamma_{a\beta}^\alpha(x)\sigma^\beta(x)\right)\mathsf{s}^a(x)\bar{e}_\alpha$$

and similarly, for $\bar{\sigma} = \sigma^\alpha \bar{e}_\alpha$,

$$\bar{\nabla}_{\mathsf{s}}\bar{\sigma} = [h\mathsf{s}, v\bar{\sigma}]_v = \left(\rho^i_a \frac{\partial \sigma^\alpha}{\partial x^i} + \Gamma^\alpha_{a\beta}(x)\sigma^\beta(x)\right)\mathsf{s}^a(x)\bar{e}_\alpha. \tag{49}$$

Corollary 1. *If* $(e, \mathsf{v}) \in \pi^*\mathsf{V}$ *and* $\mathsf{s} \in Sec(\tau)$, $\sigma \in Sec(\pi)$ *are sections passing through* v *and* e *respectively, then we have the following relation*

$$h(e, \mathsf{v}) = T\sigma(\varrho(\mathsf{v})) - [h\mathsf{s}, v\sigma](e). \tag{50}$$

Proof. It was shown in [25] that a pair of operators having the properties of covariant derivatives $(\nabla, \bar{\nabla})$ uniquely define an affine ϱ-connection on π. The brackets $[h\mathsf{s}, v\sigma]_v$ and $[h\mathsf{s}, v\bar{\sigma}]_v$ constitute such a pair (as shown above) and according to [25], the right-hand side of (50) would then define the associated affine ϱ-connection. *A priori*, however, there is no reason why this would be the h we started from. But the proof of proposition 5 precisely guarantees now that it must be the h we started from, and hence we have (50). □

There is of course a similar formula for $\bar{h}$, which reads

$$\bar{h}(\bar{e}, \mathsf{v}) = T\bar{\sigma}(\varrho(\mathsf{v})) - [h\mathsf{s}, v\bar{\sigma}](\bar{e}). \tag{51}$$

As an immediate benefit of the formulae (45), we can obtain an explicit defining relation now for the extension of the operators $(\nabla, \bar{\nabla})$ to a covariant derivative $\tilde{\nabla}$ on $\mathrm{Sec}(\tilde{\pi})$. Each $\tilde{\sigma} \in \mathrm{Sec}(\tilde{\pi})$ is either of the form $f\sigma$, with $f \in C^\infty(M)$ and $\sigma \in \mathrm{Sec}(\pi)$, or of the form $\bar{\sigma}$, for some $\bar{\sigma} \in \mathrm{Sec}(\bar{\pi})$. Then, $\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma}$ is defined in [25] by one or other of the following relations

$$\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma} = f\nabla_{\mathsf{s}}\sigma + \varrho(\mathsf{s})(f)\sigma \qquad \text{or} \qquad \tilde{\nabla}_{\mathsf{s}}\tilde{\sigma} = \bar{\nabla}_{\mathsf{s}}\bar{\sigma}. \tag{52}$$

Corollary 2. *A unifying formula for the computation of* $\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma}$ *is given by*

$$\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma} = [h\mathsf{s}, v\tilde{\sigma}]_v + \varrho(\mathsf{s})(\langle\tilde{\sigma}, e^0\rangle)\mathcal{I}. \tag{53}$$

Proof. In the first case we have $\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma} = f[h\mathsf{s}, v\sigma]_v + \varrho(\mathsf{s})(f)\sigma$. Using the properties that σ, regarded as a section of $\pi^*\tilde{\pi}$, can be written as $\sigma = \vartheta(\sigma) + \mathcal{I}$, and that $(v(\sigma))_v = \vartheta(\sigma)$, this expression can be rewritten as $\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma} = [h\mathsf{s}, v(f\sigma)]_v + \varrho(\mathsf{s})(f)\mathcal{I}$, which is of the form (53) since $\langle\tilde{\sigma}, e^0\rangle = f$ in this case. In the second case, we have $\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma} = [h\mathsf{s}, v\bar{\sigma}]_v$, which is immediately of the right form since $\langle\tilde{\sigma}, e^0\rangle = 0$ now. □

The representation (53) of $\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma}$ is exactly the decomposition (2) of $\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma}$, regarded as section of $\pi^*\tilde{\pi}$. One should not forget, of course, that such a decomposition somehow conceals part of the information in case the section under consideration, as is the case with $\tilde{\nabla}_{\mathsf{s}}\tilde{\sigma}$ here, is basic, in the sense that it is actually a section of $\tilde{\pi} : \tilde{E} \to M$.

4. Generalized connections of Berwald type

Within the framework of the classical theory of connections on a tangent bundle $\tau_M : TM \to M$ (or more generally a vector bundle over *M*), it is well known that an arbitrary horizontal distribution or non-linear connection has a kind of linearization [29]. For the case of the tangent bundle, for example, this induced linear connection can be interpreted in different equivalent ways: as a linear connection on $T(TM) \to TM$ (see, e.g., [28]), or as a linear connection on the vertical bundle $V(TM) \to TM$ (see, e.g., [1]), or perhaps most efficiently as a connection on the pullback bundle $\tau_M^*TM \to TM$. An interesting geometrical characterization of this so-called Berwald-type connection, in its pullback bundle version, was given by Crampin in [6].

Our generalization to a time-dependent set-up on jet bundles [24] revealed that there is a certain liberty in fixing the time component of the connection, though two particular choices come forward in a rather natural way via a direct defining relation of the covariant derivative. This kind of gauge freedom in fixing the connection has everything to do with the affine nature of the first-jet bundle. We shall now explore to what extent a ϱ-connection on an affine bundle, in the general picture of section 2, has a kind of induced linearization, and we intend to unravel in that process the origin of the two specific choices for fixing the connection, as described in [24].

The case of our time-dependent model in [24] fits within the present scheme as follows: $\mathsf{V} = TM$, ϱ is the identity and E is the first-jet bundle of $M \to \mathbb{R}$. The induced linear connection then is a connection on the pullback bundle $\pi^*\tilde{\pi}$, i.e. a covariant derivative operator $\nabla_\xi X$, where ξ is a vector field on E and X a section of $\pi^*\tilde{\pi}$. To define $\nabla_\xi X$, it suffices to specify separately the action of horizontal and vertical vector fields, where 'horizontality' is defined of course via the non-linear connection one starts from. In the more general situation of a ϱ-connection, however, horizontality of vector fields on E is not an unambiguous notion, in the sense that $\operatorname{Im} h$ may not provide a full complement of the set of vertical vectors and may even have a non-empty intersection with this set (see [4]). As said in section 2, we do have a direct complement for the vertical sections of $\pi^1 : T^\varrho E \to E$. So the right way to look here for a linear connection on $\pi^*\tilde{\pi}$ is as a ϱ^1-connection.

The linear ϱ^1-connection on $\pi^*\tilde{\pi}$ will actually be generated by an affine ϱ^1-connection on $\pi^*E \to E$. However, as long as we let $\mathsf{V} \to M$ be any vector bundle, not related to $E \to M$ and without the additional structure of a Lie algebroid, there is no bracket of sections of τ or π^1 available. We should, therefore, not expect to discover easily direct defining relations. Instead, we shall approach the problem of detecting corresponding ϱ^1-connections on π^*E via their covariant derivative operators, for which we will use the results of proposition 5 as one of the sources of inspiration. Following the lead of Crampin's approach in [6], the other source of inspiration should come from understanding the details of possible rules of parallel transport, into which subject we will enter now first.

Recall that the concept of parallel transport in E, i.e. the construction of the horizontal lift c_e^h of a ϱ-admissible curve in V, exists for any ϱ-connection h. If h is affine, we know that for horizontal lifts which start at e and $e_1 = e + \bar{e}$ at an initial time a, we have at any later time b that $c_{e+\bar{e}}^h(b) = c_e^h(b) + c_{\bar{e}}^{\bar{h}}(b)$. In addition, $v(e, \bar{e})$ identifies the couple $(e, \bar{e}) \in \pi^*\bar{E}$ with a vertical tangent vector to E, and we have seen that the evolution to the vector $v\big(c_e^h, c_{\bar{e}}^{\bar{h}}\big)$ is just Lie transport along c_e^h. If h is *not* affine, Lie transport of a vertical vector along c_e^h still exists and one could somehow reverse the order of thinking to use that for defining an affine action on fibres of E. To be specific, writing $Y_{e,\bar{e}}(a) = v(e, \bar{e})$ for the initial vertical vector and considering its Lie transport, defined as before by $Y_{e,\bar{e}}(u) = T\phi_{u,a}^h(Y_{e,\bar{e}}(a))$, we get the following related actions on $\pi^*\bar{E}$ and π^*E:

$$(e, \bar{e}) \mapsto \big(c_e^h, p_{\bar{E}}((Y_{e,\bar{e}})_v)\big) \qquad (e, e_1) \mapsto \big(c_e^h, c_e^h + p_{\bar{E}}((Y_{e,e_1-e})_v)\big). \tag{54}$$

We will refer to this as the *affine action on π^*E by Lie transport along horizontal curves*. Obviously, when h is not affine, the image of (e, e_1) under this affine action will not be $\big(c_e^h, c_{e_1}^h\big)$.

The question which arises now is whether there are natural ways also to define an affine action on π^*E along vertical curves, i.e. curves in a fixed fibre E_m of E. Let c_e^v denote an arbitrary curve through e in the fibre E_m ($m = \pi(e)$). It projects onto the constant curve $c_m : u \mapsto c_m(u) = m$ in M. A curve in V which has the same projection (and actually is ϱ-admissible) can be taken to be $\mathsf{o}_m : u \mapsto \mathsf{o}_m(u) = (m, \mathsf{o}_m)$. $\dot{c}_e^v$ is a curve in TE which projects onto c_e^v and has the property $T\pi\big(\dot{c}_e^v\big) = 0$. By analogy with earlier constructions, we define a new curve $\dot{c}_e^V$ in $T^\varrho E$, determined by

$$\dot{c}_e^V := \big(\mathsf{o}_m, \dot{c}_e^v\big). \tag{55}$$

Obviously, by construction, we have that $\pi^1 \circ \dot{c}_e^V = c_e^v$ and $\varrho^1 \circ \dot{c}_e^V = \dot{c}_e^v$, so that $\dot{c}_e^V$ is ϱ^1-admissible.

Let us now address the problem of defining a transport rule in π^*E along curves c_e^v. Remember that for the horizontal curves, we described such a transport rule by looking first at the way vertical tangent vectors can be transported. For the transport of vertical tangent vectors within a fixed fibre, the usual procedure is to take a simple translation (this is sometimes called *complete parallelism*). Thus, starting from a point $(e, e_1) \in \pi^*E$, with which we want to associate first a vertical tangent vector, we think of (e, e_1) as belonging to $\pi^*\tilde{E}$ and consider $v(e, e_1) = v(e, \vartheta_e(e_1)) = v(e, e_1 - e)$. Its parallel translate along a curve c_e^v is $v\big(c_e^v, e_1 - e\big)$ which can be identified with $\big(c_e^v, e_1 - e\big) \in \pi^*\bar{E}$. But it makes sense to associate with this a new element of π^*E as well, in exactly the same way as we did for horizontal curves. We thus arrive at the following action on $\pi^*\bar{E}$ and π^*E

$$(e, \bar{e}) \mapsto \big(c_e^v, \bar{e}\big) \qquad (e, e_1) \mapsto \big(c_e^v, c_e^v + e_1 - e\big). \tag{56}$$

It could be described as a *vertical affine action by translation in* $\pi^*\bar{E}$.

There is, however, another way of transporting points in π^*E along a curve of type c_e^v, which is in fact the most obvious one if one insists on having a link with a transport rule of vertical tangent vectors via the vertical lift operator. It is obtained by looking at the action

$$(e, \bar{e}) \mapsto \big(c_e^v, \bar{e}\big) \qquad (e, e_1) \mapsto \big(c_e^v, e_1\big) \tag{57}$$

and could be termed as a *vertical affine action by translation in* π^*E.

Given an arbitrary ϱ-connection h on the affine bundle $\pi : E \to M$, we now want to construct an induced ϱ^1-connection h^1 on the affine bundle $\pi^*\pi : \pi^*E \to E$ through the identification of suitable covariant derivative operators D and $\bar{\mathrm{D}}$. That is to say, we should give a meaning to things like $\mathrm{D}_{\mathcal{Z}}X$ and $\bar{\mathrm{D}}_{\mathcal{Z}}\bar{X}$, for $\mathcal{Z} \in \mathrm{Sec}(\pi^1)$, $X \in \mathrm{Sec}(\pi^*\pi)$, $\bar{X} \in \mathrm{Sec}(\pi^*\bar{\pi})$. As explained in section 2, every $\mathcal{Z}$ has a unique decomposition in the form $\mathcal{Z} = \mathsf{X}^H + \bar{Y}^V$, with $\mathsf{X} \in \mathrm{Sec}(\pi^*\tau)$, $\bar{Y} \in \mathrm{Sec}(\pi^*\bar{\pi})$. These in turn are finitely generated (over $C^\infty(E)$) by *basic sections*, i.e. sections of τ and of $\bar{\pi}$, respectively. The same is true for the sections X or $\bar{X}$ on which $\mathrm{D}_{\mathcal{Z}}$ and $\bar{\mathrm{D}}_{\mathcal{Z}}$ operate. This means that, for starting the construction of covariant derivatives, we must think of a defining relation for $\mathrm{D}_{\mathsf{s}^H}\sigma$, $\mathrm{D}_{\bar{\eta}^V}\sigma$, $\bar{\mathrm{D}}_{\mathsf{s}^H}\bar{\sigma}$, $\bar{\mathrm{D}}_{\bar{\eta}^V}\bar{\sigma}$, with $\mathsf{s} \in \mathrm{Sec}(\tau)$, $\sigma \in \mathrm{Sec}(\pi)$, $\bar{\eta}, \bar{\sigma} \in \mathrm{Sec}(\bar{\pi})$. The expectation is, since we look for a D and $\bar{\mathrm{D}}$, that h^1, as a kind of linearization of h, will be an affine connection and so, in the particular case that the given h is affine, it should essentially reproduce a copy of itself. Therefore, the first idea which presents itself is to set

$$\mathrm{D}_{\mathsf{s}^H}\sigma = [h\mathsf{s}, v\sigma]_v \qquad \bar{\mathrm{D}}_{\mathsf{s}^H}\bar{\sigma} = [h\mathsf{s}, v\bar{\sigma}]_v \qquad \mathrm{D}_{\bar{\eta}^V}\sigma = \bar{\mathrm{D}}_{\bar{\eta}^V}\bar{\sigma} = 0. \tag{58}$$

The first point in the proof of proposition 5 did not rely on the assumption of h being affine, so we know that these formulae at least are consistent with respect to the module structure over $C^\infty(M)$. We then extend the range of the operators D and $\bar{\mathrm{D}}$ in the obvious way, by the following three rules: for every $F \in C^\infty(E)$, we put

$$\mathrm{D}_{F\mathsf{s}^H}\sigma = F\mathrm{D}_{\mathsf{s}^H}\sigma = F[h\mathsf{s}, v\sigma]_v \qquad \bar{\mathrm{D}}_{F\mathsf{s}^H}\bar{\sigma} = F\bar{\mathrm{D}}_{\mathsf{s}^H}\bar{\sigma} = F[h\mathsf{s}, v\bar{\sigma}]_v \tag{59}$$

$$\mathrm{D}_{F\bar{\eta}^V}\sigma = \bar{\mathrm{D}}_{F\bar{\eta}^V}\bar{\sigma} = 0 \tag{60}$$

which suffices to know what $\mathrm{D}_{\mathcal{Z}}\sigma$ and $\bar{\mathrm{D}}_{\mathcal{Z}}\bar{\sigma}$ mean for arbitrary $\mathcal{Z} \in \mathrm{Sec}(\pi^1)$, and finally we put

$$\bar{\mathrm{D}}_{\mathcal{Z}}(F\bar{\sigma}) = F\bar{\mathrm{D}}_{\mathcal{Z}}\bar{\sigma} + \varrho^1(\mathcal{Z})(F)\bar{\sigma} \tag{61}$$

$$\mathrm{D}_{\mathcal{Z}}(\sigma + F\bar{\sigma}) = \mathrm{D}_{\mathcal{Z}}\sigma + F\bar{\mathrm{D}}_{\mathcal{Z}}\bar{\sigma} + \varrho^1(\mathcal{Z})(F)\bar{\sigma} \tag{62}$$

which suffices to give a meaning to all $\mathrm{D}_{\mathcal{Z}}X$ and $\bar{\mathrm{D}}_{\mathcal{Z}}\bar{X}$. Our operators satisfy by construction all the necessary requirements for defining an affine ϱ^1-connection h^1.

It is worthwhile to observe that for the covariant derivatives of general $X \in \mathrm{Sec}(\pi^*\pi)$ and $\bar{X} \in \mathrm{Sec}(\pi^*\bar{\pi})$, we still have an explicit formula at our disposal when $\mathcal{Z}$ is of the form s^H, with s basic. This follows from the fact that $\varrho^1(\mathsf{s}^H) = h(\mathsf{s})$, so that

$$\begin{aligned}\mathrm{D}_{\mathsf{s}^H}(\sigma + F\bar{\sigma}) &= [h\mathsf{s}, v\sigma]_v + F[h\mathsf{s}, v\bar{\sigma}]_v + \varrho^1(\mathsf{s}^H)(F)\bar{\sigma}\\ &= [h\mathsf{s}, v(\sigma + F\bar{\sigma})]_v\end{aligned} \tag{63}$$

and likewise for $\mathrm{D}_{\mathsf{s}^H}\bar{X}$.

The next point on our agenda is to understand what parallel transport means for the affine connection $(\mathrm{D}, \bar{\mathrm{D}})$, or even better, to show that it is uniquely characterized by certain features of its parallel transport. The general idea of parallel transport is clear, of course: starting from any ϱ^1-admissible curve c^1 in $T^{\varrho}E$, its horizontal lift is a curve ψ^1 in π^*E having the same projection $\psi^1_E = c^1_E$ in E and satisfying $\mathrm{D}_{c^1}\psi^1 = 0$; image points of ψ^1 then give parallel translation by definition. Now, ψ^1 is essentially a pair of curves in E having the same projection in M, so the determination of ψ^1 is a matter of constructing a second curve in E having the same projection in M as c^1_E. It will be sufficient to focus on ϱ^1-admissible curves of the form $\dot{c}^H_e$ and $\dot{c}^V_e$, for which the corresponding projections on E are curves of the form c^h_e and c^v_e, respectively, and to consider curves ψ^1 which come from the restriction of sections of $\pi^*\pi$ to c^h_e or c^v_e. To simplify matters even further, we can use basic sections $\mathsf{s} \in \mathrm{Sec}(\tau)$ to generate horizontal curves, because we know from lemma 1 that the integral curves of $h(\mathsf{s}) \in \mathcal{X}(E)$ are horizontal lifts. Vertical curves, of course, can be generated as integral curves of vertical vector fields.

Proposition 6. *Let $\mathsf{s} \in \mathrm{Sec}(\tau)$, $\bar{Y} \in \mathrm{Sec}(\pi^*\bar{\pi})$ be arbitrary. Denote the integral curves of $h(\mathsf{s})$ and $v\bar{Y}$ through a point e by c^h_e and c^v_e and consider their lifts to ϱ^1-admissible curves $\dot{c}^H_e$ and $\dot{c}^V_e$ in $T^{\varrho}E$. $(\mathrm{D}, \bar{\mathrm{D}})$ is the unique affine ϱ^1-connection on $\pi^*\pi$ with the properties:*

(i) Parallel transport along $\dot{c}^H_e$ is the affine action on π^*E by Lie transport along horizontal curves.

(ii) Parallel transport along $\dot{c}^V_e$ is the vertical affine action by translation in π^*E.

Proof. Recall that $\mathsf{s}^H \in \mathrm{Sec}(\pi^1)$ is defined at each $e \in E$ by $\mathsf{s}^H(e) = (\mathsf{s}(\pi(e)), h(\mathsf{s})(e))$, so that at each point along an integral curve c^h_e of $h(\mathsf{s})$, we have

$$\mathsf{s}^H\big(c^h_e(u)\big) = \big(\mathsf{s} \circ \pi \circ c^h_e(u), \dot{c}^h_e(u)\big) = \dot{c}^H_e(u). \tag{64}$$

Let now X be an arbitrary section of $\pi^*\pi$ and put $\psi^1(u) = X\big(c^h_e(u)\big)$, which defines a curve in π^*E projecting onto c^h_e in E. We have

$$\big(\mathrm{D}_{\dot{c}^H_e}\psi^1\big)(u) = \mathrm{D}_{\dot{c}^H_e(u)}X = \mathrm{D}_{\mathsf{s}^H(c^h_e(u))}X = (\mathrm{D}_{\mathsf{s}^H}X)\big(c^h_e(u)\big). \tag{65}$$

If such curve is required to govern parallel transport in π^*E, we must have $\big(\mathrm{D}_{\dot{c}^H_e}\psi^1\big)(u) = 0, \forall u$. This implies that $\forall \mathsf{s} \in \mathrm{Sec}(\tau), \forall X \in \mathrm{Sec}(\pi^*\pi)$, $\mathrm{D}_{\mathsf{s}^H}X$ should be zero along integral curves of $h(\mathsf{s}) \in \mathcal{X}(E)$. By the observation leading to the explicit formula (63) for $\mathrm{D}_{\mathsf{s}^H}X$ and with $v(X)\big(c^h_e(u)\big) = v(\psi^1(u))$, which defines a vertical vector field along c^h_e, this requirement is further equivalent to $\mathcal{L}_{h(\mathsf{s})}v(\psi^1) = 0$, which is precisely the characterization of Lie transport. The same arguments apply to $\bar{\mathrm{D}}$ and show that our $(\mathrm{D}, \bar{\mathrm{D}})$ has the property (i).

With $\bar{Y} \in \mathrm{Sec}(\pi^*\bar{\pi})$, $\bar{Y}^V \in \mathrm{Sec}(\pi^1)$ is such that $\varrho^1(\bar{Y}^V)$ is a vertical vector field on E. Hence, its integral curves are curves of the form c^v_e in a fixed fibre $E_{\pi(e)}$ and we have from (55):

$$\bar{Y}^V\big(c^v_e(u)\big) = \big(\mathsf{o}_{\pi(e)}, \dot{c}^v_e(u)\big) = \dot{c}^V_e(u). \tag{66}$$

Let X again be an arbitrary section of $\pi^*\pi$ and put this time $\psi^1(u) = X\big(c_e^v(u)\big)$, which defines a curve in $E \times_M E$ projecting onto c_e^v for its first component. We wish to show that if $\psi^1(u)$ rules parallel transport along $\dot{c}_e^V$, it is necessarily a curve which is constant in its second component. We have

$$\big(\mathrm{D}_{\dot{c}_e^V}\psi^1\big)(u) = \mathrm{D}_{\dot{c}_e^V(u)}X = \mathrm{D}_{\bar{Y}^V(c_e^v(u))}X = (\mathrm{D}_{\bar{Y}^V}X)\big(c_e^v(u)\big). \tag{67}$$

There is no explicit formula available for $\mathrm{D}_{\bar{Y}^V}X$. However, X is locally of the form $X = \sigma + F_i\bar{\sigma}_i$, with σ, $\bar{\sigma}_i$ basic sections and $F_i \in C^\infty(E)$. It then follows that $\mathrm{D}_{\bar{Y}^V}X = \varrho^1(\bar{Y}^V)(F_i)\bar{\sigma}_i$ and the requirement $\mathrm{D}_{\bar{Y}^V}X\big(c_e^v(u)\big) = 0$ implies that the F_i must be first integrals of $\varrho^1(\bar{Y}^V)$. In turn this means that the value $X\big(c_e^v(u)\big)$ is constant. This way we see that the affine connection $(\mathrm{D}, \bar{\mathrm{D}})$ also has property (ii).

That properties (i) and (ii) uniquely fix the connection is easy to see, because the above arguments show that they impose in particular that $\mathrm{D}_{\mathsf{s}^H}\sigma = [h\mathsf{s}, v\sigma]_v$ and $\mathrm{D}_{\bar{\eta}^V}\sigma = 0$ (and similarly for $\bar{\mathrm{D}}$), for basic s, σ and $\bar{\eta}$. And these are exactly the defining relations (58) from which our couple $(\mathrm{D}, \bar{\mathrm{D}})$ was constructed. □

We have seen earlier on that there is a second interesting transport rule along vertical curves and would like to discover now what modifications to the affine connection must be made to have this other rule as vertical parallel transport. We are referring here to the action (56) for which the curve starting at some $(e, e_1) \in E_m \times E_m$ is of the form

$$u \overset{\psi^1}{\mapsto} \big(c_e^v(u), e_1 + c_e^v(u) - e\big) = \big(c_e^v(u), c_e^v(u) + e_1 - e\big). \tag{68}$$

Now, it is easy to identify a section of $\pi^*\pi$ which along c_e^v coincides with this curve. Indeed, choosing a basic section $\bar{\sigma} \in \mathrm{Sec}(\bar{\pi})$ which at $m = \pi(e)$ coincides with $e_1 - e$, we are simply looking at the restriction to c_e^v of $\mathcal{I} + \bar{\sigma}$, where $\mathcal{I}$ here denotes the identity map on E.

Let $(\hat{\mathrm{D}}, \hat{\bar{\mathrm{D}}})$ denote the affine connection we are looking for now and which clearly will coincide with $(\mathrm{D}, \bar{\mathrm{D}})$ for its 'horizontal action'. If, as before, $\bar{Y} \in \mathrm{Sec}(\pi^*\bar{\pi})$ generates the vertical vector field $v\bar{Y}$ whose integral curves are the c_e^v, the above ψ^1 will produce parallel transport, provided we have

$$\big(\hat{\mathrm{D}}_{\dot{c}_e^V}\psi^1\big)(u) = \hat{\mathrm{D}}_{\bar{Y}^V}(\mathcal{I} + \bar{\sigma})\big(c_e^v(u)\big) = 0. \tag{69}$$

Since this must hold for each $\bar{Y}^V$ and, for every fixed $\bar{Y}^V$ also for all $\bar{\sigma}$, this is equivalent to requiring that $\hat{\mathrm{D}}_{\bar{Y}^V}\mathcal{I} = 0$ and $\hat{\bar{\mathrm{D}}}_{\bar{Y}^V}\bar{\sigma} = 0, \forall \bar{Y}, \bar{\sigma}$. In fact, in view of the linearity in $\bar{Y}$, we actually obtain the conditions

$$\hat{\mathrm{D}}_{\bar{\eta}^V}\mathcal{I} = 0 \qquad \text{and} \qquad \hat{\bar{\mathrm{D}}}_{\bar{\eta}^V}\bar{\sigma} = 0 \tag{70}$$

for all basic $\bar{\sigma}$ and $\bar{\eta}$. It is interesting to characterize this completely by properties on basic sections, because the extension to a full affine connection on $\pi^*\pi$ then follows automatically. If σ is an arbitrary basic section of $\pi^*\pi$, it can be decomposed (see (2)) in the form $\sigma = \mathcal{I} + \vartheta(\sigma)$, whereby $\vartheta(\sigma)(e) = (e, \sigma(\pi(e)) - e)$. Clearly, $\sigma(\pi(e)) - e$, as an element of $\bar{E}$, has components which are linear functions of the fibre coordinates of e, in such a way that when acted upon by the vector field $\varrho^1(\bar{\eta}^V)$, we will obtain $-\bar{\eta}$. It follows that $\hat{\bar{\mathrm{D}}}_{\bar{\eta}^V}\vartheta(\sigma) = -\bar{\eta}$ and therefore that

$$\hat{\mathrm{D}}_{\bar{\eta}^V}\mathcal{I} = 0 \qquad \Longleftrightarrow \qquad \hat{\mathrm{D}}_{\bar{\eta}^V}\sigma = -\bar{\eta} \qquad \forall \sigma \in \mathrm{Sec}(\pi). \tag{71}$$

This way, we have detected an alternative way for defining an affine ϱ^1-connection $(\hat{\mathrm{D}}, \hat{\bar{\mathrm{D}}})$ on $\pi^*\pi$. Compared to (58), its defining relations are

$$\hat{\mathrm{D}}_{\mathsf{s}^H}\sigma = [h\mathsf{s}, v\sigma]_v \qquad \hat{\bar{\mathrm{D}}}_{\mathsf{s}^H}\bar{\sigma} = [h\mathsf{s}, v\bar{\sigma}]_v \qquad \hat{\mathrm{D}}_{\bar{\eta}^V}\sigma = -\bar{\eta} \qquad \hat{\bar{\mathrm{D}}}_{\bar{\eta}^V}\bar{\sigma} = 0. \tag{72}$$

We can further immediately draw the following conclusion about its characterization.

Proposition 7. *With the same premises as in proposition 6,* $(\hat{\mathrm{D}}, \hat{\bar{\mathrm{D}}})$ *is the unique affine* ϱ^1*-connection on* $\pi^*\pi$ *with the properties:*

(i) Parallel transport along $\dot{c}_e^H$ *is the affine action on* $\pi^* E$ *by Lie transport along horizontal curves.*
(ii) Parallel transport along $\dot{c}_e^V$ *is the vertical affine action by translation in* $\pi^*\bar{E}$.

We will refer to the connections $(\mathrm{D}, \bar{\mathrm{D}})$ and $(\hat{\mathrm{D}}, \hat{\bar{\mathrm{D}}})$, as well as their extensions $\tilde{\mathrm{D}}$ and $\tilde{\hat{\mathrm{D}}}$ as *Berwald-type connections*. For completeness, we list their defining relations here in coordinates.

$$\begin{aligned} \mathrm{D}_{\mathcal{H}_a} e_0 &= \left(\Gamma_a^\gamma - y^\beta \frac{\partial \Gamma_a^\gamma}{\partial y^\beta}\right)\bar{e}_\gamma \qquad && \mathrm{D}_{\mathcal{V}_\alpha} e_0 = 0 \\ \bar{\mathrm{D}}_{\mathcal{H}_a} \bar{e}_\beta &= \frac{\partial \Gamma_a^\gamma}{\partial y^\beta}\bar{e}_\gamma && \bar{\mathrm{D}}_{\mathcal{V}_\alpha} \bar{e}_\beta = 0 \end{aligned} \tag{73}$$

and

$$\begin{aligned} \hat{\mathrm{D}}_{\mathcal{H}_a} e_0 &= \left(\Gamma_a^\gamma - y^\beta \frac{\partial \Gamma_a^\gamma}{\partial y^\beta}\right)\bar{e}_\gamma \qquad && \hat{\mathrm{D}}_{\mathcal{V}_\alpha} e_0 = -\bar{e}_\alpha \\ \hat{\bar{\mathrm{D}}}_{\mathcal{H}_a} \bar{e}_\beta &= \frac{\partial \Gamma_a^\gamma}{\partial y^\beta}\bar{e}_\gamma && \hat{\bar{\mathrm{D}}}_{\mathcal{V}_\alpha} \bar{e}_\beta = 0. \end{aligned} \tag{74}$$

5. The case of affine Lie algebroids and the canonical connection associated with a pseudo-Sode

It is now time to relate the new and quite general results of the preceding sections to some of our earlier work. Recall that our interest in affine bundles comes in the first place from the geometrical study of time-dependent second-order equations and the analysis of Berwald-type connections in that context [24]. Secondly, once the potential relevance for applications of Lagrangian systems on Lie algebroids became apparent, we were led to explore a time-dependent generalization of such systems and thus arrived at the introduction and study of affine Lie algebroids [22, 27]. Note that Lagrangian systems on algebroids are particular cases of pseudo-Sodes, but the concept of a pseudo-Sode in itself, strictly speaking, does not require the full structure of a Lie algebroid.

So, let us start by looking at pseudo-Sodes on the affine bundle E, which are essentially vector fields with the property that all the integral curves are ρ-admissible. In saying that, we are in fact assuming that the anchor map has E in its domain. In this section, therefore, the starting point is that we have an affine bundle map $\rho : E \to TM$ at our disposal with associated linear map $\bar{\rho} : \bar{E} \to TM$. These maps can be extended to a linear map $\tilde{\rho} : \tilde{E} \to TM$ as follows: for every $\tilde{e} \in \tilde{E}_m$, making a choice of an element $e \in E_m$, we have a representation of the form $\tilde{e} = \lambda e + \bar{e}$ and define $\tilde{\rho}(\tilde{e})$ by

$$\tilde{\rho}(\tilde{e}) = \lambda\rho(e) + \bar{\rho}(\bar{e}). \tag{75}$$

One easily verifies that this construction does not depend on the choice of e. In coordinates

$$\rho : (x^i, y^\alpha) \mapsto \left(\rho_\alpha^i(x) y^\alpha + \rho_0^i(x)\right)\frac{\partial}{\partial x^i} \qquad \tilde{\rho} : (x^i, y^0, y^\alpha) \mapsto \left(\rho_\alpha^i y^\alpha + \rho_0^i y^0\right)\frac{\partial}{\partial x^i}. \tag{76}$$

In what follows $\tilde{\rho}$ plays the role of the anchor map ϱ we had before. This means in particular that the vector bundle $\tau : \mathsf{V} \to M$ from now on is taken to be the bundle $\tilde{\pi} : \tilde{E} \to M$. Now, pseudo-Sodes can be regarded also as sections of the prolonged bundle $\pi^1 : T^{\tilde{\rho}}E \to E$ (rather than as vector fields on E). The difference in interpretation is easy to understand from the basic constructions explained in section 2. Indeed, we have seen that there is a natural vertical lift operator $v : \mathrm{Sec}(\pi^*\tilde{\pi}) \to \mathcal{X}(E)$, which extends to an operator ${}^V : \mathrm{Sec}(\pi^*\tilde{\pi}) \to \mathrm{Sec}(\pi^1)$ via (9). Combining this vertical lift with the projection $j : T^{\tilde{\rho}}E \to \pi^*\tilde{E}$, gives rise to the map $S = {}^V \circ j$, called the *vertical endomorphism on* $\mathrm{Sec}(\pi^1)$. For $\mathcal{Z} = \zeta^0\mathcal{X}_0 + \zeta^\alpha\mathcal{X}_\alpha + Z^\alpha\mathcal{V}_\alpha$, we have

$$S(\mathcal{Z}) = (\zeta^\alpha - \zeta^0 y^\alpha)\mathcal{V}_\alpha. \tag{77}$$

An elegant definition of the concept of pseudo-Sode then goes as follows.

Definition 2. *A pseudo-*Sode *is a section* Γ *of* π^1 *such that* $S(\Gamma) = 0$ *and* $\langle\Gamma, \mathcal{X}^0\rangle = 1$.

In coordinates, Γ is of the form

$$\Gamma = \mathcal{X}_0 + y^\alpha\mathcal{X}_\alpha + f^\alpha\mathcal{V}_\alpha. \tag{78}$$

It is not immediately clear whether a pseudo-Sode comes with a canonically associated (non-linear) $\tilde{\rho}$-connection in this general setting. However, as mentioned already in [26], the construction of a connection becomes quite obvious when we have the additional structure of a Lie algebroid.

So, assume now we have an affine Lie algebroid structure on π, which can most conveniently be seen as a Lie algebroid on $\tilde{\pi}$ with anchor $\tilde{\rho}$, which is such that the bracket of two sections of π (regarded as sections of $\tilde{\pi}$) is a section of $\bar{\pi}$ (also considered as a section of $\tilde{\pi}$). In coordinates, there exist structure functions $C^\gamma_{\alpha\beta}$ and $C^\gamma_{0\beta}$ on M such that

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}(x)e_\gamma \qquad \text{and} \qquad [e_0, e_\beta] = C^\gamma_{0\beta}(x)e_\gamma. \tag{79}$$

We have shown in [22] that such a Lie algebroid on $\tilde{\pi}$ can be prolonged to a Lie algebroid on $\tilde{\pi}^1$ with anchor $\tilde{\rho}^1$. In coordinates

$$\begin{aligned} &[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C^\gamma_{\alpha\beta}\mathcal{X}_\gamma \qquad &&[\mathcal{X}_0, \mathcal{X}_\beta] = C^\gamma_{0\beta}\mathcal{X}_\gamma \qquad &&[\mathcal{V}_\alpha, \mathcal{X}_\beta] = 0 \\ &[\mathcal{X}_0, \mathcal{V}_\beta] = 0 &&[\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0. \end{aligned} \tag{80}$$

The Lie algebroid structure provides us with an exterior derivative; we use the standard notation d_Γ for the commutator $[i_\Gamma, d]$, which plays the role of Lie derivative and extends, as a degree zero derivation, to tensor fields of any type.

Now, one way of pinning down a $\tilde{\rho}$-connection on π consists in identifying its horizontal projector P_H (and then $P_V = I - P_H$).

Proposition 8. *If* Γ *is a pseudo-*Sode *on an affine Lie algebroid* π*, then the operator*

$$P_H = \tfrac{1}{2}(I - d_\Gamma S + \mathcal{X}^0 \otimes \Gamma) \tag{81}$$

defines a horizontal projector on Sec(π^1) *and hence a* $\tilde{\rho}$*-connection on* π.

Proof. The proof follows the lines of the classical one for time-dependent mechanics (see [5] or [9]). Since it was largely omitted in [26], we give a brief sketch of one possibility to proceed here. For $\tilde{\sigma} \in \mathrm{Sec}(\tilde{\pi})$, define the horizontal lift $\tilde{\sigma}^H \in \mathrm{Sec}(\pi^1)$ by

$$\tilde{\sigma}^H = \tfrac{1}{2}(\tilde{\sigma}^C + \langle\tilde{\sigma}, e^0\rangle\Gamma - [\Gamma, \tilde{\sigma}^V]) \tag{82}$$

where $\tilde{\sigma}^C$ is the complete lift, as defined in [22]. It is easy to see that this behaves tensorially for multiplication by basic functions and that $\tilde{\sigma}^H$ projects onto σ. Hence, extending the horizontal lift to $\mathrm{Sec}(\pi^*\tilde{\pi})$ by imposing linearity for multiplication by functions on E, we obtain a splitting of the short exact sequence (10) for the present situation. This in fact concludes the proof of the existence of a $\tilde{\rho}$-connection, but it is interesting to verify further the explicit formula for P_H. One can, for example, compute the Lie algebroid brackets $[\Gamma, \bar{\sigma}^V]$

and $[\Gamma, \bar{\sigma}^H]$ for $\bar{\sigma} \in \mathrm{Sec}(\bar{\pi})$, from which it then easily follows (using also the properties $S(\bar{\sigma}^H) = \bar{\sigma}^V$ and $S(\bar{\sigma}^V) = 0$), that $d_\Gamma S(\bar{\sigma}^V) = \bar{\sigma}^V$, $d_\Gamma S(\bar{\sigma}^H) = -\bar{\sigma}^H$ and $d_\Gamma S(\Gamma) = 0$. The verification that P_H is a projection operator and that $P_H(\tilde{\sigma}^H) = \tilde{\sigma}^H$ then is immediate. □

The connection coefficients of the pseudo-SODE connection are given by

$$\Gamma^\alpha_0 = -f^\alpha + \frac{1}{2} y^\beta \left(\frac{\partial f^\alpha}{\partial y^\beta} + C^\alpha_{0\beta} \right) = -f^\alpha - y^\beta \Gamma^\alpha_\beta \tag{83}$$

$$\Gamma^\alpha_\beta = -\frac{1}{2} \left(\frac{\partial f^\alpha}{\partial y^\beta} + y^\gamma C^\alpha_{\gamma\beta} + C^\alpha_{0\beta} \right). \tag{84}$$

Briefly, the particular case of a Lagrangian system on the affine Lie algebroid π is obtained as follows. Let L be a function on E and consider the 1-form $\theta_L = \mathrm{d}L \circ S + L\mathcal{X}^0$. If $\omega_L = \mathrm{d}\theta_L$ has maximal rank at every point, i.e. when L is said to be regular, there exists a unique pseudo-SODE such that $i_\Gamma \omega_L = 0$. In that case, the functions f^α which determine the connection coefficients are given by

$$f^\alpha = g^{\alpha\beta} \left(\rho^i_\beta \frac{\partial L}{\partial x^i} + \left(C^\gamma_{\mu\beta} y^\mu + C^\gamma_{0\beta} \right) \frac{\partial L}{\partial y^\gamma} - \left(\rho^i_0 + \rho^i_\mu y^\mu \right) \frac{\partial^2 L}{\partial x^i \partial y^\beta} \right) \tag{85}$$

where $(g^{\alpha\beta})$ stands for the inverse matrix of $(g_{\alpha\beta}) = \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \right)$.

Having now seen sufficient reasons to pay particular attention to the case of affine Lie algebroids, we come back to the construction of Berwald-type connections associated with arbitrary $\tilde{\rho}$-connections. So assume we have a $\tilde{\rho}$-connection on the affine Lie algebroid π (not necessarily of pseudo-SODE type). As explained in section 2, it is then appropriate to work with the adapted basis $\{\mathcal{H}_a, \mathcal{V}_\alpha\}$ for $\mathrm{Sec}(\pi^1)$, rather than the 'coordinate basis' $\{\mathcal{X}_a, \mathcal{V}_\alpha\}$ (here the index a stands for either 0 or α). The following bracket relations then become useful:

$$\begin{aligned} [\mathcal{H}_a, \mathcal{V}_\alpha] &= \frac{\partial \Gamma^\delta_a}{\partial y^\alpha} \mathcal{V}_\delta \\ [\mathcal{H}_a, \mathcal{H}_b] &= C^\delta_{ab} \mathcal{H}_\delta + \left(C^\delta_{ab} \Gamma^\gamma_\delta + \tilde{\rho}^1(\mathcal{H}_b)\left(\Gamma^\gamma_a\right) - \tilde{\rho}^1(\mathcal{H}_a)\left(\Gamma^\gamma_b\right) \right) \mathcal{V}_\gamma. \end{aligned} \tag{86}$$

It will further be appropriate to write now $_H$ for the projection $T^{\tilde{\rho}}E \to \pi^*\tilde{E}$ and likewise define the map $_V : T^{\tilde{\rho}}E \to \pi^*\bar{E} \subset \pi^*\tilde{E}$ by: $\mathcal{Z}_V = (\tilde{\rho}^1(P_V \mathcal{Z}))_v$. The reason is that this will bring us in line with notation used in [6, 24] to which the next proposition strongly relates. Combining the horizontal and vertical lift operations with the direct sum decomposition (3) of $\mathrm{Sec}(\pi^*\tilde{\pi})$, it is more convenient now to think of the following threefold decomposition of $\mathrm{Sec}(\pi^1)$:

$$\mathrm{Sec}(\pi^1) = \langle \mathcal{I}^H \rangle \oplus \mathrm{Sec}(\pi^*\bar{\pi})^H \oplus \mathrm{Sec}(\pi^*\bar{\pi})^V. \tag{87}$$

Note that, in the particular case of a pseudo-SODE connection, we have $\mathcal{I}^H = \Gamma$.

We know that any $\tilde{\rho}$-connection generates Berwald-type connections. The strong point of the next result, however, is that if we assume that π is an affine Lie algebroid, there is a direct defining formula for the two Berwald-type connections discussed in the preceding section.

Proposition 9. *If the affine bundle π carries an affine Lie algebroid structure, the Berwald-type connections $\tilde{\mathrm{D}}$ and $\tilde{\tilde{\mathrm{D}}}$ are determined by the following direct formulae:*

$$\tilde{\mathrm{D}}_{\mathcal{Z}} \tilde{X} = [P_H \mathcal{Z}, \bar{X}^V]_V + [P_V \mathcal{Z}, \bar{X}^H]_H + \tilde{\rho}^1(P_H \mathcal{Z})(\langle \tilde{X}, e^0 \rangle) \mathcal{I} \tag{88}$$

$$\tilde{\tilde{\mathrm{D}}}_{\mathcal{Z}} \tilde{X} = [P_H \mathcal{Z}, \bar{X}^V]_V + [P_V \mathcal{Z}, \bar{X}^H]_H + \tilde{\rho}^1 \mathcal{Z}(\langle \tilde{X}, e^0 \rangle) \mathcal{I} \tag{89}$$

with $\bar{X} := \tilde{X} - \langle \tilde{X}, e^0 \rangle \mathcal{I}$.

Proof. Using the properties ${}_V \circ P_H = 0$, ${}_H \circ P_V = 0$, $h \circ {}_H = \tilde{\rho}^1 \circ P_H$, it is easy to verify that the above expressions satisfy the appropriate rules when the arguments are multiplied by a function on *E*. Hence, both operators define a linear $\tilde{\rho}^1$-connection on the vector bundle $\pi^*\tilde{\pi}$. Next, we verify that this connection comes from an affine $\tilde{\rho}^1$ connection on $\pi^*\pi$. For that, according to a result in [25], it is necessary and sufficient that e^0 (here regarded as the basic section of $\pi^*\tilde{\pi}$) is parallel. We have

$$(\tilde{\mathrm{D}}_{\mathcal{Z}} e^0)(\tilde{X}) = \tilde{\rho}^1 \mathcal{Z}(\langle \tilde{X}, e^0 \rangle) - \langle \tilde{\mathrm{D}}_{\mathcal{Z}} \tilde{X}, e^0 \rangle \tag{90}$$

and similarly for $\tilde{\bar{\mathrm{D}}}$. In the case of $\tilde{\mathrm{D}}$, we have

$$\langle \tilde{\mathrm{D}}_{\mathcal{Z}} \tilde{X}, e^0 \rangle = \langle [P_V \mathcal{Z}, \tilde{X}^H]_H, e^0 \rangle + \tilde{\rho}^1 (P_H \mathcal{Z})(\langle \tilde{X}, e^0 \rangle). \tag{91}$$

Making use of the first of the bracket relations (86), it is straightforward to verify that the first term on the right is equal to $\tilde{\rho}^1(P_V \mathcal{Z})(\langle \tilde{X}, e^0 \rangle)$, so that the sum of both terms indeed makes the right-hand side of (90) vanish. The computation for $\tilde{\bar{\mathrm{D}}}$ is similar.

It remains now to check that the restrictions to $\mathrm{Sec}(\pi^*\pi)$ and $\mathrm{Sec}(\pi^*\bar{\pi})$ of (88) and (89) verify, respectively, the defining relations (58) and (72) for $(\mathrm{D}, \bar{\mathrm{D}})$ and $(\hat{\mathrm{D}}, \bar{\hat{\mathrm{D}}})$. If we take $\mathcal{Z} = \tilde{\sigma}^H$ and $\tilde{X} = \eta$, for basic $\tilde{\sigma} \in \mathrm{Sec}(\tilde{\pi})$ and $\eta \in \mathrm{Sec}(\pi)$, then we know from proposition 5 that the bracket $[h\tilde{\sigma}, v\eta]$ is vertical in *TE*. As a consequence $(0, [h\tilde{\sigma}, v\eta])$ is vertical in $T^{\tilde{\rho}}E$. But this is precisely $[\tilde{\sigma}^H, \eta^V]$, because the bracket of the two projectable sections $\tilde{\sigma}^H$ and η^V is by construction (see [22]) the section $([\tilde{\sigma}, 0], [\tilde{\rho}^1\tilde{\sigma}^H, \tilde{\rho}^1\eta^V])$ of π^1. Therefore, $\mathrm{D}_{\tilde{\sigma}^H}\eta = [\tilde{\sigma}^H, \eta^V]_V = (\tilde{\rho}^1(P_V[\tilde{\sigma}^H, \bar{\eta}^V]))_v = (\tilde{\rho}^1[\tilde{\sigma}^H, \bar{\eta}^V])_v = [h\tilde{\sigma}, v\bar{\eta}]_v$, where the Lie algebra homomorphism provided by the anchor map $\tilde{\rho}^1$ has been used. Similar arguments apply for the other operators $\bar{\mathrm{D}}$, $\hat{\mathrm{D}}$ and $\bar{\hat{\mathrm{D}}}$ when $\mathcal{Z}$ is horizontal. In remains to look at the case $\mathcal{Z} = \bar{\sigma}^V$ $(\bar{\sigma} \in \mathrm{Sec}(\bar{\pi}))$. Since $[\bar{\sigma}^V, \bar{\eta}^H]$ is vertical, it follows that $\bar{\mathrm{D}}_{\bar{\sigma}^V}\bar{\eta} = \bar{\hat{\mathrm{D}}}_{\bar{\sigma}^V}\bar{\eta} = 0$. For $\tilde{X} = \eta$, since then $\langle \eta, e^0 \rangle = 1$, we find for the first connection $\mathrm{D}_{\bar{\sigma}^V}\eta = 0$. For the second connection, it suffices to check (see (70)) that $\hat{\mathrm{D}}_{\bar{\sigma}^V}\mathcal{I} = 0$, and this is trivial. □

6. Conclusions

Two main objectives have been attained in this paper: we have unravelled the mechanism by which a generalized connection over an anchored bundle leads to a linearized connection over an appropriate prolonged anchored bundle; we have at the same time focussed on the special features of connections on an affine bundle, in general, and on an affine Lie algebroid in particular. The latter subject is a completion of the work we started in [25]. But it also ties up with the first issue, as a generalization of the study of Berwald-type connections in [24], where we dealt, so to speak, with the prototype of an affine Lie algebroid, namely the first-jet extension of a bundle fibred over $\mathbb{R}$, this being the geometrical arena for time-dependent mechanics.

What are such Berwald-type connections good for? The covariant derivative operators associated with (classical) Berwald-type connections are those which are at the heart of the theory of derivations of forms along the tangent (or first-jet) bundle projection, initiated in [19, 20]. These operators have proved to be very useful tools in a number of applications concerning qualitative features of SODEs. We mention, for example, the characterization of linearizability [7, 18] and of separability [3, 21] of SODEs; the inverse problem of Lagrangian mechanics [10]; the study of Jacobi fields and Raychaudury's equation [13]. There is little doubt that there are similar applications ahead for the qualitative study of pseudo-SODEs on Lie algebroids.

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